BMS INSTITUTE OF TECHNOLOGY AND MANAGEMENT

YELAHANKA – BANGALORE - 64

DEPARTMENT OF ELECTRONICS & TELECOMMUNICATION ENGINEERING

Course Name:	**Digital Signal Processing**
Course Code:	**18EC52**
Semester :	**5th semester**
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CONTENTS

DISCRETE FOURIER TRANSFORMS (DFT)

Frequency domain sampling & reconstruction of discrete time signals. DFT as a linear transformation, its relationship with other transforms. 6 Hrs

Introduction to DSP

* A signal is a function i.e used to describe an observed physical variable of a physical process.

* It is an abstract mathematical description of an observation.

eg. Image signal, ECG (electrocardiogram) about condition of heart, EEG (electroencephalogram) about activity of brain, speech signal, its mathematical description consists of infinite no of sinusoids of diff' freq's & can be written as

$$s(t) = \sum_{i=-\infty}^{\infty} A_i(t) \sin[\Omega_i(t) + \theta_i(t)]$$

amplitude, freq, phase angle

* System: A system may also be defined as a physical device that performs an operation on a signal

eg. speech signals are generated by forcing air thro' the vocal cords. That system consists of vocal cords & vocal tract (vocal cavity)

eg. filter : used to reduce noise & interference
It performs some ~~pu~~ operation(s) on the signal which has the effect of reducing (filtering) noise & interference from the desired information bearing signal

signal processing : deals with manipulation or modification of signal so that it results in more desirable or interpretable form.

(or)

The action of changing one or more features (parameters) of a signal according to a predetermin requirements

* parameters → freq, amplitude, phase etc
* The signal which undergoes such a process is known as input signal
* The entity which performs this processing is system
* The processed signal is o/p signal

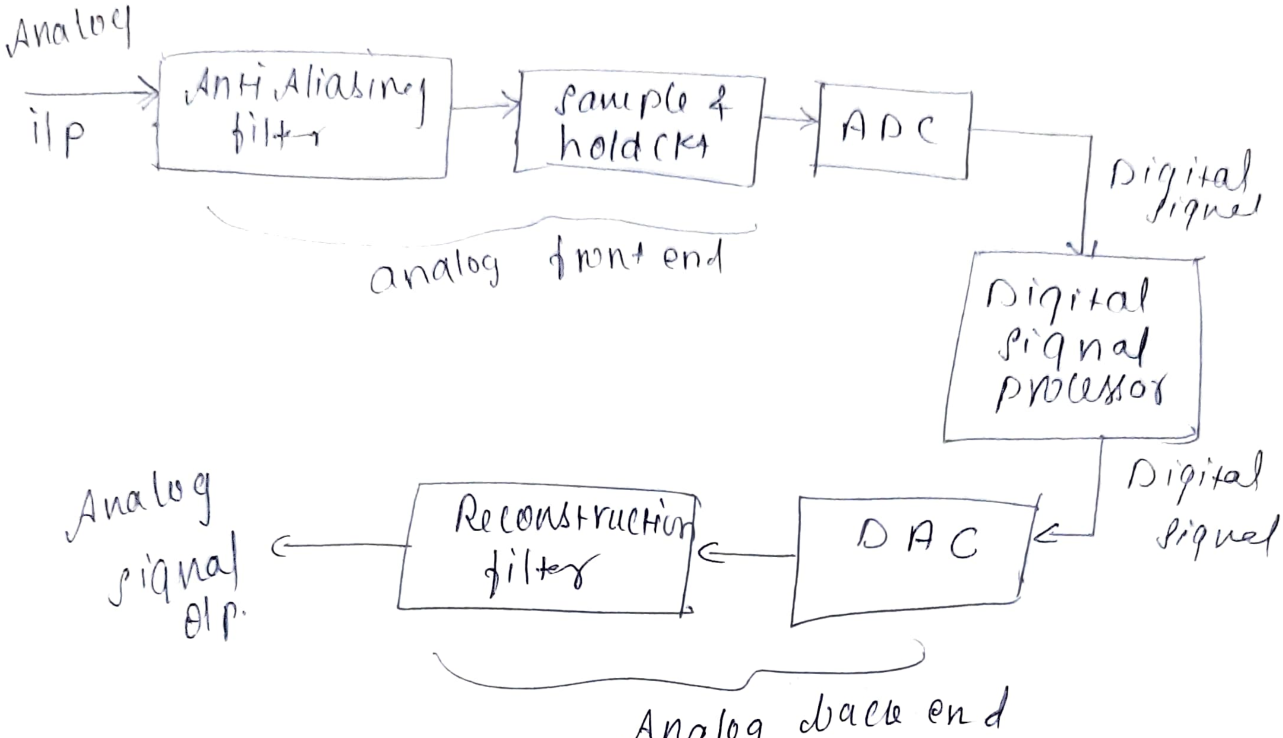
eg of signal processing : Amplification, filtering, modn, etc

Analog

All these uses transistor, FET's, Op-amp.

* A platform for signal processing can be analog system, where we can design an analog system using design technique, test it & redesign until an acceptable performance is met

* There is an another approach called Digital signal processing where the info is represented in digital format.



* The antialiasing filter, an analog LPF is used to band limit the i/p analog signal to the required freq ~~component~~ range & prevent freq components beyond this range from appearing as aliases of the i/p signal in the sampled spectrum

* The sample & hold ckt → discretes the analog i/p signal wrt time
 · the analog i/p signal is picked up at discrete intervals in time

* The ADC maps /assigns binary value to each sample appearing at the i/p chosen from a set of finite value. we get a digital signal discretized both wrt time & amplitude

* The digital signal so obtained is processed by employing DSP techniques & the o/p is another seqn of binary no's which is converted to analog form using DAC

* The LPF at o/p removes the undesired high freq noise & gives out the desired analog signal

Advantages of Digital over Analog Signal Processing

1) digital pgmmable system allows flexibility in reconfiguring the digital signal processing opern's simply by changing the pgm

Reconfiguration of an analog system usually implies a redesign of the H/w followed by testing & verification to see that it operates properly

2) digital system provides much better control of accuracy requirements

3) digital signals are easily stored on magnetic media (tape or disk) w/o loss of signal

4) DSP method also allows for the implementation of more sophisticated signal processing algorithms

(5) It is difficult to perform precise mathematical operns on signals in analog form but the same operns can be implemented on a digital computer using s/w's ③

(6) digital implementation of signal processing system is cheaper than its analog.

(7) Adaptable to low freq signal processing where analog processors would require very large passive elements like inductors & capacitors

Limitation

* system complexity incs ∵ need of conversion of real-life (time) analog signals to digital & processed digital signal back to analog
* Limited BW due to constraint on sampling rate.

Applications

* consumer electronic applns - TV, music synthesizer,

1. The Discrete Fourier Transforms

It's properties & Applications

* Frequency analysis of discrete-time signals is usually & most conveniently performed on a digital signal processor [i.e. may be a general purpose digital computer or specially designed digital H(ω)].

* To perform frequency Analysis on a discrete-time signal {x(n)}, we convert the time-domain sequence to an equivalent freq-domain representation. such a representation is given by the Fourier Transform X(ω) of the sequence {x(n)}

* However, X(ω) is a ~~constar~~ continuous fun of freq & ∴ it is not a computationally convenient representation of the sequence {x(n)}

* Here we consider the representation of a sequence {x(n)} by samples of its spectrum X(ω). Such a freq-domain representation leads to the discrete Fourier Transform [DFT], which is a powerfull computational tool for performing freq Analysis of discrete-time signals

(Ref.: Prokais)

(1.1) Frequency Domain Sampling: The Discrete Fourier Transform

Before we introduce DFT, we consider the sampling of the FT of an aperiodic discrete-time seq^n. Thus we establish the relationship bet' the sampled FT & DFT.

Frequency Domain Sampling & Reconstruction of Discrete-Time Signals

~~WKT~~, let us consider an aperiodic discrete-time signal x(n) with fourier transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \longrightarrow (1)$$

x(n) → discrete-time signals

ω → freq & is continuous fun from 0 to 2π.

* This means that x(n) is discrete & its spectrum X(ω) is continuous. Such a continuous fun^s cannot be evaluated on a digital processor. Since only ~~digita~~ discrete signals can be evaluated.
* So to overcome this problem of digital processing, the spectrum X(ω) is sampled uniformly (peri... in freq at a spacing of δω radians bet' successive... with period of 2π,
* Since X(ω) is periodic from 0 to 2π. The samples are taken from 0 to 2π & the spacing bet' successive samples will be

$$\delta\omega = \frac{2\pi}{N}.$$

∴ for convenience we take N equidista... samples in the interval

$0 \leq \omega \leq 2\pi$ with spacing

∴ substituting $\omega = \frac{2\pi}{N}k$ in eq ①, we get

(2) ⑤

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi}{N}kn} \longrightarrow ②$$

where $k = 0, 1, 2, \ldots N-1$

: index of samples.

& $X(\omega)$ ~~can~~ is calculated only at discrete values.

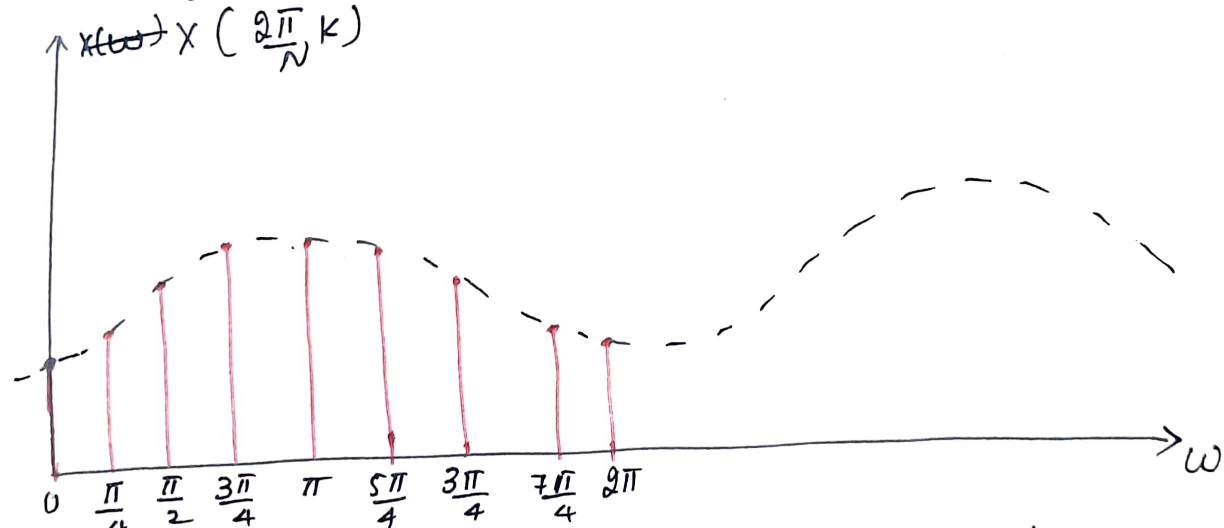
* for eg let $N = 8$. samples are taken over a period of 2π & $X(\omega)$ will be calculated at

$$\omega = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{4} \text{ \& } \frac{7\pi}{4}$$

& the samples are addressed as

$k = 0, 1, 2, \ldots 7$

by substituting these values in eq ②, the sampled spectrum obtained is shown in fig (1.1) below.



$k = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$

fig (1.1) freq - domain sampling of the fourier - transforms from eq ②, 'n' varies from $-\infty$ to $+\infty$, let us divide this summation into individual summations ~~containg~~ containing only 'N' samples of $x(n)$

i.e,

$$X\left(\frac{2\pi}{N}k\right) = \cdots + \sum_{n=-N}^{-1} x(n) e^{-j\frac{2\pi kn}{N}} + \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}$$

$$+ \sum_{n=N}^{2N-1} x(n) e^{-j\frac{2\pi kn}{N}} + \cdots$$

The above ~~indid~~ individual summations can be represented as

$$X\left(\frac{2\pi}{N}k\right) = \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n) e^{-j\frac{2\pi kn}{N}}$$

let us change $n \rightarrow n - lN$ of the inner summation hence the limits will be at $n=0$ to $N-1$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n-lN) e^{-j\frac{2\pi k(n-lN)}{N}}$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n-lN) e^{-j\frac{2\pi kn}{N}} e^{j2\pi kl}$$

here $e^{j2\pi kl} = 1$ always

$$X\left(\frac{2\pi}{N}k\right) = \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n-lN) e^{-\frac{j2\pi kn}{N}}$$

let us interchange the order of the summation. we obtain,

~~n = n - lN~~

$= \cancel{n - lN} - lN$

(i) $n = lN$

$n = n - lN$

$= ln - lN$

$\boxed{n = 0}$

(ii) $lN + N - 1$

$n = n - lN$

$= lN + N - 1 - lN$

$\boxed{= N-1}$

$e^{+\theta} = \cos\theta + j\sin\theta$

$e^{\pm j2\pi k}$

$= \cos 2\pi k + j\sin 2\pi k$

$= 1 + 0$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} x(n-lN)\right] e^{-j2\pi kn/N} \longrightarrow (3)$$

$K = 0 \ldots N-1$

$$= \sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi kn/N} \longrightarrow (4)$$

here $K = 0, 1, \ldots N-1$

$$\& \quad x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN) \longrightarrow (4.1)$$

$$\ldots + x(n+2N) + x(n+N) + x(n) + x(n-N) + x(n-2N) + \ldots$$

This means that $x_p(n)$ is a periodic repetition of $x(n)$ with the period of 'N' samples.

* Let us consider some arbitrary non-periodic signal $x(n)$. It contains 'L' samples & is shown in fig (1.2a)

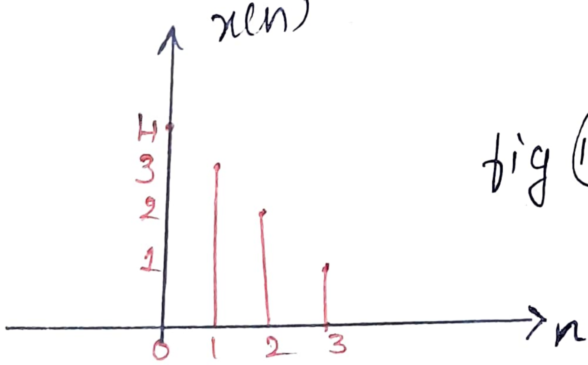


fig (1.2a) An original signal $x(n)$ of length 'L'

case-1

(i) $N > L$, No Aliasing:

Now let us prepare the signal $x_p(n)$ which is obtained by periodic repetition of $x(n)$

Let the period 'N' be greater that 'L'

for eg. let $N = 6$ & the w/f is shown in fig (1.2b). here at $n = 4 \& 5$, the sample values are zero

because the signal repeats at $n=6, 12 \ldots$ etc &
hence for $N > L$, there is no Aliasing

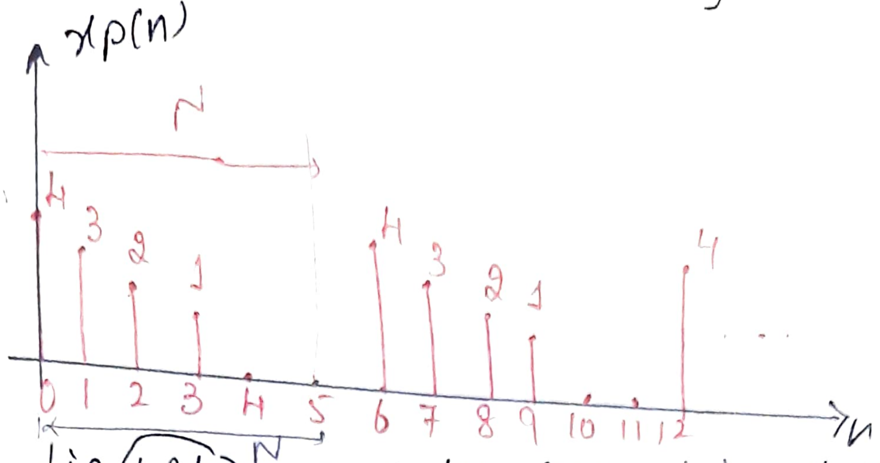


fig (1.2b) periodic repetition of $x(n)$ with period $N > L$

case (ii) :· $N < L$, Aliasing

Now let us consider $N < L$, for eg $N = 3$ the w/f is shown in fig (1.2c).

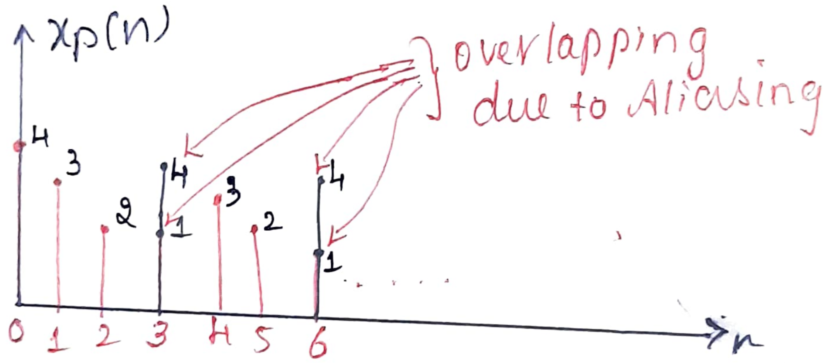


fig (1.2c) periodic repetition of $x(n)$ with periodic $N < L$.

here from the w/f we observe that since $N < L$, the 2 samples are overlapping at $n = 3, 6 \ldots$ etc. This is Aliasing hence it isn't possible to obtain $x(n)$ from $xp(n)$.

∴ To avoid Aliasing in time-domain, the no of samples in freq spectrum must be greater than no of samples in time-domain seqn ie $N \geq L$.

Reconstruction:

WKT $x_p(n)$ is periodic with period 'N'

It can be expressed by discrete Fourier series as

$$x_p(n) = \sum_{K=0}^{N-1} C_K e^{j\frac{2\pi}{N}Kn} \rightarrow (5)$$

where $n = 0, 1, \ldots N-1$

C_K = Fourier coefficient & can be expressed as

$$C_K = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi}{N}Kn} \rightarrow (6)$$

$K = 0, 1, \ldots N-1$

WKT from eq (4) $X(\frac{2\pi}{N}K)$ is

$$X(\frac{2\pi}{N}K) = \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi}{N}Kn} \rightarrow (4)$$

$\therefore$ comparing eq (4) & (6) we get

$$C(K) = \frac{1}{N} X(\frac{2\pi}{N}K) \rightarrow (7)$$

Subtituting (7) in (5)

$$x_p(n) = \sum_{K=0}^{N-1} \frac{1}{N} X(\frac{2\pi}{N}K) e^{j\frac{2\pi}{N}Kn} \rightarrow (8)$$

$n = 0, 1, \ldots N-1$

This eqn (8) provides the reconstruction of periodic signal $x_p(n)$ from the samples of spectrum $X(\omega)$

* However it doesn't imply that we can recover $X(\omega)$ or $x(n)$ from the sam

* to accomplish this we need to consider the relationship bet' $x_p(n)$ + $x(n)$

* since $x_p(n)$ is periodic extension of $x(n)$ as in eq (4.1)

i.e, $$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

* It is clear than $x(n)$ can be recovered from $x_p(n)$ if there is no aliasing in time domain, as illustrated in the fig 1.2.

* ~~from the fig we observe~~ when

* we consider a finite-duration seq^n $x(n)$ which is non zero in the interval $0 \le n \le L-1$

* we observe that when $N \ge L$

$$x(n) = x_p(n), \quad 0 \le n \le N-1$$

so that $x(n)$ can be recovered from $x_p(n)$ without ambiguity

* if $N < L$ it is not possible to recover $x(n)$ from its periodic extension due - time domain Aliasing.

(35)

* Thus we conclude that the spectrum of an aperiodic discrete-time signal with finite duration 'L' can be exactly recovered from its samples at freqs

$$\omega_k = \frac{2\pi k}{N} \quad \text{if} \quad N \geq L.$$

$$x(n) = \begin{cases} x_p(n), & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases} \rightarrow (7)$$

& finally $X(\omega)$ can be computed eq (1)

as $$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

(1.2) Discrete Fourier Transform [DFT]

If $x(n)$ has a finite durn of length $L \leq N$, then $x_p(n)$ is periodic repetition of $x(n)$

where $x_p(n) = \begin{cases} x(n), & 0 \leq n \leq L-1 \\ 0, & L \leq n \leq N-1 \end{cases}$

[defined for one period of N]

* consider eq (11) i.e

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi kn}{N}}$$

from fig (1.2b) we have shown that if no of samples in $x(n)$ are less than N then there is no Aliasing. If we calculate the above eq for $x(n)$, then we can write

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{L-1} x(n)\, e^{-j\frac{2\pi}{N}kn} \longrightarrow (9)$$

WKT $N > L$, to avoid Aliasing in time domain hence the upper limit of the summation can be made as $N-1$.

$\therefore$ eq (9) becomes

DFT:- $$\boxed{X(k) = X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} \qquad k = 0, 1 \ldots N-1} \longrightarrow (10)$$

Note:- $X\left(\frac{2\pi}{N}k\right)$ is written as $X(k)$. $\therefore$ the values of $X\left(\frac{2\pi}{N}k\right)$ are addressed by 'k' only.

eq (10) is called as Discrete Fourier Transform (DFT)

Now consider eq (8)

$$x_p(n) = \sum_{K=0}^{N-1} \frac{1}{N} X\left(\frac{2\pi}{N}K\right) e^{\frac{j2\pi Kn}{N}}$$

if we evaluate the above eq^n for $n = 0, 1, \ldots N-1$ then $x_p(n) = x(n)$

$$\boxed{\text{IDFT}:- \; x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{\frac{j2\pi Kn}{N}} \quad n = 0, 1, \ldots N-1} \rightarrow (11)$$

The above eq^n gives the original seq $x(n)$ from its DFT. Hence it is called as Inverse Discrete Fourier Transforms [IDFT].

DFT as a linear transformation:-

Let us define $W_N = e^{-j\frac{2\pi}{N}} \rightarrow (12)$

= Twidle factor.

hence DFT & IDFT eqn can be written as

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{Kn}, \quad K = 0, 1, \ldots N-1 \rightarrow (13)$$

&

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{-Kn}, \quad n = 0, 1, \ldots N-1 \rightarrow (14)$$

So eq (13) & (14) can be denoted symbolically as follows:

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{-Kn} \xleftrightarrow{\text{DFT}} X(K) = \sum_{n=0}^{N-1} x(n) W_N^{Kn}$$

Let us ~~den~~ represent seq $x(n)$ as vector x_N of 'N' samples i.e.

$$x_N = \begin{array}{c} n=0 \\ n=1 \\ \vdots \\ n=N-1 \end{array} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}_{N\times 1} \longrightarrow (15)$$

& $X(K)$ can be represented as a vector X_N of 'N' samples i.e.

$$X_N = \begin{array}{c} K=0 \\ K=1 \\ \vdots \\ K=N-1 \end{array} \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}_{N\times 1} \longrightarrow (16)$$

The values W_N^{kn} can be represented as a matrix $[W_N]$ of size $N \times N$ as follows:

$$[W_N] = \begin{array}{l} W_N^{kn}\Big|_{K=0,n=0} = W_N^0 \\ W_N^{kn}\Big|_{K=1,n=0} = W_N^0 \\ W_N^{kn}\Big|_{K=2,n=0} = W_N^0 \\ \vdots \\ W_N^{kn}\Big|_{K=N-1,n=0} = W_N^0 \end{array}$$

$$[W_N] = \begin{array}{c|cc} & n=0 & n=1 \\ \hline K=0 & W_N^{kn}\big|_{K=0,n=0} = W_N^0 & W_N^{kn}\big|_{K=0,n=1} = W_N^0 \\ K=1 & W_N^{kn}\big|_{K=1,n=0} = W_N^0 & W_N^{kn}\big|_{K=1,n=1} = W_N^1 \\ K=2 & W_N^{kn}\big|_{K=2,n=0} = W_N^0 & W_N^{kn}\big|_{K=2,n=1} = W_N^2 \\ \vdots & \vdots & \vdots \\ K=N-1 & W_N^{kn}\big|_{K=N-1,n=0} = W_N^0 & W_N^{kn}\big|_{K=N-1,n=1} = W_N^{N-1} \end{array}$$

$$\begin{array}{cccc} n=2 & & n=N-1 & \\ W_N^{kn}\big|_{K=0,n=2} = W_N^0 & \cdots & W_N^{kn}\big|_{K=0,n=N-1} = W_N^0 \\ W_N^{kn}\big|_{K=1,n=2} = W_N^2 & \cdots & W_N^{kn}\big|_{K=1,n=N-1} = W_N^{N-1} \\ W_N^{kn}\big|_{K=2,n=2} = W_N^4 & \cdots & W_N^{kn}\big|_{K=2,n=N-1} = W_N^{2(N-1)} \\ \vdots & & \vdots \\ W_N^{kn}\big|_{K=N-1,n=2} = W_N^{2(N-1)} & \cdots & W_N^{kn}\big|_{K=N-1,n=N-1} = W_N^{(N-1)(N-1)} \end{array} \quad N \times N$$

$$\therefore [W_N] = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \rightarrow (17)$$

here the individual elements are written as W_N^{kn} with 'K' rows & 'n' columns

Then N-point DFT of eq (13) can be represented in the matrix form as

$$\boxed{X_N = [W_N]\, x_N} \longrightarrow (18)$$

Ill'y, IDFT of eq (14) can be represented in the matrix form as

$$\boxed{x_N = \frac{1}{N}\,[W_N^*]\, X_N} \longrightarrow (19)$$

where $W_N^* \rightarrow$ complex conjugate of W_N.

~~comparing eqn's (18) & (19)~~

$W_N \rightarrow$ matrix of linear trans

* We observe that W_N is a symmetric matrix, if we assume that inverse of W_N exists then eq (18) can be written as

$$\boxed{x_N = W_N^{-1} X_N} \longrightarrow (20)$$

* comparing eq (19) & (20),

$$\frac{1}{W_N} = \frac{1}{N} W_N^*$$

$$W_N^{-1} = \frac{1}{N} W_N^*$$

(or) $$\boxed{W_N W_N^* = N \cdot I_N} \longrightarrow (21).$$

where I_N is an $N \times N$ identity matrix

Let us see the values of W_N for the foll values:

(i) $N = 8$

WKT $W_N = e^{-j\frac{2\pi}{N}}$

with $N = 8$ $\quad W_N = e^{-j\frac{2\pi}{8}} = e^{-j\frac{2\pi}{4}}$

Then $W_N^K = e^{-j\frac{2\pi}{N}K}$

$\therefore W_8^K = e^{-j\frac{2\pi}{8}K}$

$= e^{-j\frac{\pi}{4}K}$ where $K = 0, 1, 2 \ldots N-1$

(i) $\underline{\underline{K=0}}$ $\quad W_8^0 = e^0 = 1$

(i) $K = 1$ $\quad W_8^1 = e^{\frac{-j\pi}{4}} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

(ii) $K = 2$ $\quad W_8^2 = e^{-j\frac{2\pi}{4}} = \cos\pi/2 - j\sin\pi/2 = 0 - j1$

$= e^{-j\pi/2}$ $\qquad = -j$

(iii) $K = 3$ $\quad W_8^3 = e^{-j\frac{3\pi}{4}} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

(iv) $K = 4$ $\quad W_8^4 = e^{-j\frac{4\pi}{4}} = \cos\pi - j\sin\pi = -1$

$= e^{-j\pi}$

$\boxed{W_8^4 = -W_8^0}$

(v) $K = 5$ $\quad W_8^5 = e^{\frac{-j5\pi}{4}} = \cos\frac{5\pi}{4} - j\sin\frac{5\pi}{4} = \frac{-1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

$\boxed{W_8^5 = -W_8^1}$

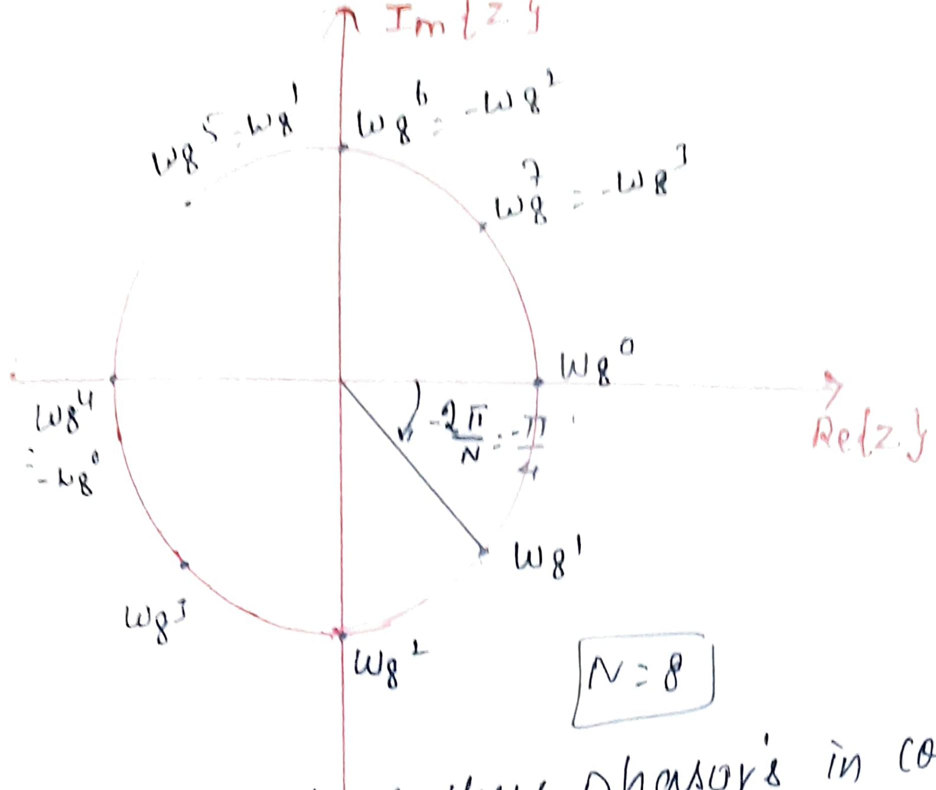
(vi) $K = 6$ $\quad W_8^6 = e^{-j\frac{6\pi}{4}} = e^{-j\frac{3\pi}{2}} = \cos\frac{3\pi}{2} \cancel{+} - j\sin\frac{3\pi}{2}$

$\boxed{W_8^6 = -W_8^2}$ $\qquad = 0 + j = j$

(vii) $K = 7$ $\quad W_8^7 = e^{-j\frac{7\pi}{4}} = \cos\frac{7\pi}{4} - j\sin\frac{7\pi}{4}$

$= 1/\sqrt{2} + j\,1/\sqrt{2}$

$\boxed{W_8^7 = -W_8^3}$



$\boxed{N = 8}$

the fig above shows these phasor's in complexplane against unit circle

(ii) <u>N = 4</u>

$W_N^K = W_4^K$, $K = 0, 1, 2, 3$.

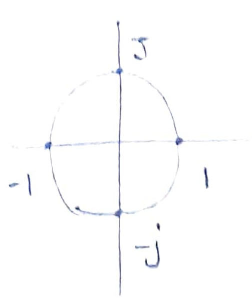
$W_N^K = e^{-j\frac{2\pi}{N}K}$ $\quad W_4^K = e^{-j\frac{2\pi}{4}K} = e^{-j\frac{\pi}{2}K}$

$K = 0 \quad W_4^0 = e^0 = 1$

$K = 1 \quad W_4^1 = e^{-j\frac{\pi}{2}} = \cos\pi/2 - j\sin\pi/2 = -j$

$K = 2 \quad W_4^2 = e^{-j\pi} = \cos\pi - j\sin\pi = -1$

$K = 3 \quad W_4^3 = e^{-j\frac{3\pi}{2}} = \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} = +j$



DFT of some standard signals.

(1) Compute the N-point DFT for the foll signals (4M)

(a) $x(n) = \delta(n)$ [unit sample]

Compute the N-point DFT of foll signals
$x(n) = \delta(n)$
$x(n) = \delta(n-n_0)$
$x(n) =$

Soln:- The unit sample $\delta(n)$ is given as

$$x(n) = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases}$$

DFT is given by

$$X(K) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

Substituting for $x(n)$

$$X(K) = x(0) e^{0}$$
$$= 1$$

(b) $x(n) = \delta(n-n_0)$

$$X(K) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} \delta(n-n_0) e^{-j\frac{2\pi}{N}kn}$$

$\because \delta(n-n_0) = 1 ; n=n_0$
$= 0, n \neq n_0$

$$= \delta(n_0) e^{-j\frac{2\pi}{N}kn_0}$$

$$= e^{-j\frac{2\pi}{N}kn_0}$$

(c) ~~$a^n u(n)$~~

(c) a

(c) $x(n) = a^n$ for $0 \le n \le N-1$

$$X(K) \triangleq \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} a^n e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} \left[a e^{-j\frac{2\pi}{N}k}\right]^n$$

WKT

$$\left[\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}, \quad \alpha \ne 1 \atop = N, \quad \alpha = 1\right]$$

(or)

$$\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1-a}$$

$$X(K) = \frac{1 - \left[a e^{-j\frac{2\pi}{N}k}\right]^N}{1 - a e^{-j\frac{2\pi}{N}k}} = \frac{1 - a^N e^{-j2\pi k}}{1 - a e^{-j\frac{2\pi}{N}k}}$$

$$e^{-j2\pi k} = \cos 2\pi k - j\sin 2\pi k$$
$$= 1 \text{ always}$$

$$\therefore X(K) = \frac{1 - a^N}{1 - a e^{-j\frac{2\pi}{N}k}}$$

$$\boxed{\text{(d)} \quad x(n) = e^{j\frac{2\pi}{N}k_0 n}, \quad 0 \le n \le N-1}$$

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$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}k_0 n}\, e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi}{N}(k-k_0)}\right)^n$$

$$= \frac{1 - \left(e^{\frac{j2\pi(k-k_0)}{N}}\right)^N}{1 - e^{j\frac{2\pi}{N}(k-k_0)}}$$

$$= \frac{1 - e^{j2\pi(k-k_0)}}{1 - e^{j\frac{2\pi}{N}(k-k_0)}}$$

when $k \neq k_0$

$$1 - e^{j2\pi k}\, e^{-j2\pi k_0}$$

$$= \frac{1-1}{1 - e^{j\frac{2\pi}{N}(k-k_0)}} = 0$$

when $k = k_0$

$$X(k) = \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi}{N}(k_0-k_0)}\right)^n$$

$$= \sum_{n=0}^{N-1} 1 = N$$

$$X(k) = \begin{cases} 0, & k \neq m \\ N, & k = m \end{cases} = N \cdot \delta(k-m)$$

(2) compute 8-point DFT for the foll seqn:

$x(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$

Soln:

WKT $W_N \triangleq e^{-j\frac{2\pi}{N}}$

since $N = 8$, $\quad W_8^k = e^{-j\frac{\pi}{4}k} \quad k = 0, 1, 2, \ldots 7$

$W_8^0 = 1, \quad W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}, \quad W_8^2 = -j, \quad W_8^3 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

$W_8^4 = -1 \quad W_8^5 = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \quad W_8^6 = j \quad W_8^7 = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

$$DFT\{x(n)\} = X(K) \triangleq \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$K = 0, 1, 2 \ldots N-1$

$$\therefore X(K) = \sum_{n=0}^{7} x(n)\, W_8^{Kn}, \quad 0 \le k \le 7$$

$X(K) = 1 \cdot W_8^0 + 1 \cdot W_8^K + 1 \cdot W_8^{2K} + 1 \cdot W_8^{3k}$

$= 1 + W_8^K + W_8^{2K} + W_8^{3k}$

$X(0) = 1 + 1 + 1 + 1 = \boxed{4}$

$X(1) = 1 + W_8^1 + W_8^2 + W_8^3 = 1 + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} - j - \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

$= 1 - j\frac{1}{\sqrt{2}} - j - j\frac{1}{\sqrt{2}}$

$\boxed{X(1) = 1 - j\,2.41}$

$X(2) = 1 + W_8^2 + W_8^4 + W_8^6$

$= 1 - j - 1 + j = \boxed{0}$

$X(3) = 1 + W_8^3 + W_8^6 + W_8^9$

$= 1 + W_8^3 + W_8^6 + W_8^1$

$= 1 - \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} + j + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} = \boxed{1 - j0.41}$

$X(4) = 1 + W_8^4 + W_8^0 + W^4 = 1 - 1 + 1 - 1 = \boxed{0}$

$X(5) = 1 + W_8^5 + W_8^2 + W_8^7 = 1 - \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} - j + \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

$= \boxed{1 + j0.41}$

$X(6) = 1 + W_8^6 + W_8^4 + W_8^2 = \boxed{0}$

$X(7) = 1 + W_8^7 + W_8^6 + W_8^5 = \boxed{1 + j2.41}$

$$\boxed{X(K) = \{ \underset{\uparrow}{4}, 1 - j2.41, 0, 1 - j0.41, 0, 1 + j0.41, 0, 1 + j2.41 \}}$$

③ ③ find the 4-point DFT of the seq^n given below $x(n) = \{1, 0, 1, 0\}$
using $X(K)$ found above find its inverse using the defining eq^n.

Soln:- $N = 4$, $W_N \triangleq e^{-j\frac{2\pi}{N}}$

$W_4 = e^{-j\pi/2}$

$W_4^0 = 1$, $W_4^1 = -j$, $W_4^2 = -1$, $W_4^3 = j$

$$DFT\{x(n)\} = X(K) \triangleq \sum_{n=0}^{N-1} x(n) W_N^{Kn}$$

$$= \sum_{n=0}^{3} x(n) W_4^{Kn} \qquad 0 \le K \le 3$$

$= x(0) W_4^0 + x(1) W_4^K + x(2) W_4^{2K} + x(3) W_4^{3K}$

$= 1 + W_4^{2K}$

$= 1 + W_4^{2K}$

$X(0) = 1 + 1 = 2$

$X(1) = 1 + W_4^2 = 1 - 1 = 0$

$X(2) = 1 + W_4^4 = 1 + W_4^0 = 1 + 1 = 2$

$X(3)\ 1 + W_4^6 = 1 + W_4^2 = 1 - 1 = 0$

$X(K) = \{ \underset{\uparrow}{2}, 0, 2, 0 \}$

Note:- $W_N^{-kn} = [W_N^{kn}]^*$

$\therefore W_4^{-0} = (W_4^0)^* = 1$, $W_4^{-1} = [W_4^1]^* = j$

$W_4^{-2} = [W_4^2]^* = -1$ $W_4^{-3} [W_4^3]^* = -j$

$$x(n) = IDFT\{X(K)\} \triangleq \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{-Kn}$$

$$x(n) = \frac{1}{4} \sum_{K=0}^{3} X(K) W_4^{-Kn} \qquad n = 0, 1, 2, 3$$

$$= \frac{1}{4} \left[2 + 0 W_4^{-n} + 2 \times W_4^{-2n} + 0 \times W_4^{-3n} \right]$$

$$= \frac{1}{4} \left[2 + 2 W_4^{-2n} \right] = \frac{1}{2} \left[1 + W_4^{-2n} \right]$$

$x(0) = \frac{1}{2} [1 + W_4^0] = 1$

$x(1) = \frac{1}{2} [1 + W_4^{-2}] = 0$

$x(2) = \frac{1}{2} [1 + W_4^0] = 1$

$x(3) = \frac{1}{2} [1 + W_4^{-2}] = 0$

$x(n) = \{1, 0, 1, 0\}$

Q Compute 4-point DFT of the seq^n using matrix method. $x(n) = \{0, 1, 2, 3\}$

Sol^n:
here $N = 4$ evaluate W_4^K where $K = 0, 1, 2, 3$

$W_4^0 = 1$, $W_4^1 = -j$, $W_4^2 = -1$ & $W_4^3 = j$

$$x_N = x_4 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad X_N = ?$$

$$[W_N]_{N\times N} = [W_4] = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}_{4\times 4}$$

$W_4^0 = W_4^4 = W_4^8$

$W_4^1 = W_4^5 = W_4^9$

$W_4^2 = W_4^6 = W_4^{10}$

$W_4^3 = W_4^7 = W_4^{11}$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & 1 \\ 1 & j & -1 & -j \end{bmatrix}$$

NOW WKT

DFT $X_N \triangleq [W_N]\, x_N$

$$X_4 = [W_4]\, x_4$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & 1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$X_4 = \begin{bmatrix} 0+1+2+3 \\ 0-j-2+3j \\ 0-1+2-3 \\ 0+j-2-3j \end{bmatrix} = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

(2) Calculate ~~8~~ 8-point DFT of the foll seq^n

$x(n) = \{1, 1, 1, 1\}$

$$X(K) = \begin{bmatrix} 4 \\ 1-j(1+\sqrt{2}) \\ 0 \\ 1+j(1-\sqrt{2}) \\ 0 \\ 1-j(1-\sqrt{2}) \\ 0 \\ 1+j(1+\sqrt{2}) \end{bmatrix}$$

(5) find 4-point DFT of the seq^n

$x(n) = \cos\left(\frac{n\pi}{4}\right)$

The 1st four samples of $x(n)$ can be obtained by putting $n = 0, 1, 2, 3$

$x(0) = \cos(0) = 1$, $x(2) = \cos\left(2 \cdot \frac{\pi}{4}\right) = 0$

$x(1) = \cos(\pi/4) = 0.707$, $x(3) = \cos\left(\frac{3\pi}{4}\right) = -0.707$

The DFT of this seq^n is

$$X(K) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0.707 \\ 0 \\ -0.707 \end{bmatrix}$$

$$= \{1,\ 1-j1.414,\ 1,\ 1+j1.414\}$$

Relationship of DFT to other Transforms :-

(i) Relationship to the fourier Transform : of a non-periodic sequence

The FT of a non periodic seq^n $x(n)$ having length 'N' is given by

$$X(e^{jw}) = \sum_{n=0}^{N-1} x(n)\, e^{-jwn} \longrightarrow ①$$

where $X(e^{jw})$ is a continuous fun of w.

The discrete FT of $x(n)$ is given by

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}Kn} \longrightarrow ②$$

$K = 0, 1, 2 \ldots N-1$

comparing eq ① & ② we find that DFT of $x(n)$ is sampled version of the FT of the seq^n & is given by

$$\boxed{X(K) = X(e^{jw})\Big|_{w=\frac{2\pi}{N}K}} \longrightarrow ③$$

$K = 0, 1, 2 \ldots N-1$

(ii) Relationship to the Z-transform :-

Let us consider a seq^n $x(n)$ of ~~the finite~~ ~~durn 'N'~~ with Z-transform.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \longrightarrow ⑤.$$

with ROC that includes the until circle if $X(z)$ is sampled at 'N' equally spaced points on the unit circle $Z_K = e^{-j\frac{2\pi}{N}k}$;

i.e. $X(K) = X(z) \Big|_{z = e^{j\frac{2\pi}{N}k}}$

$$= \sum_{n=+\infty}^{+\infty} x(n)\, e^{-j\frac{2\pi}{N}kn} \longrightarrow (2)$$

eq (2) is identical to the FT $X(e^{j\omega})$ evaluated at N equally spaced freqs

$$\omega_K = \frac{2\pi}{N}k, \quad 0 \le K \le N-1$$

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let us consider a seq^n $x(n)$ of finite dur^n 'N' with Z-transform.

$$X(z) = \sum_{n=0}^{N-1} x(n)\, z^{-n} \longrightarrow (3)$$

we have

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, e^{j\frac{2\pi}{N}kn} \longrightarrow (4)$$

$$, 0 \le n \le N-1$$

substuting eq (4) in eq (3)

$$X(z) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{K=0}^{N-1} X(K)\, e^{j\frac{2\pi}{N}kn}\right] z^{-n}$$

not need

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K) \sum_{n=0}^{N-1}\left[e^{j\frac{2\pi}{N}K} z^{-1}\right]^{n}$$

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}, \; a \ne 1$$
$$= N, \; a = 1$$

$$= \cancel{\frac{1}{N}\sum_{K=0}^{N-1} X(K)\; 1 -}$$

$$X(z) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\left[\frac{1 - e^{j\frac{2\pi}{N}K.N} z^{-N}}{1 - e^{j\frac{2\pi}{N}K} z^{-1}}\right]$$

$[e^{j2\pi k} = 1]$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K)\left[\frac{1 - z^{-N}}{1 - e^{j\frac{2\pi}{N}k} z^{-1}}\right]$$

$$= \frac{1 - z^{-N}}{N}\sum_{K=0}^{N-1} \frac{X(K)}{1 - e^{j\frac{2\pi}{N}K} z^{-1}} \rightarrow (5)$$

if eq (5) is evaluated on a unit circle at 'N' equally spaced points [i.e, $z_K = e^{j\frac{2\pi}{N}K}$; $0 \le K \le N-1$]

we get the FT of the finite dur^n seq $x(n)$ in terms of its DFT

$$X(e^{jw}) = \frac{1 - e^{-jwN}}{N}\sum_{K=0}^{N-1} \frac{X(K)}{1 \; e^{-j(w - \frac{2\pi}{N}k)}} \rightarrow (6)$$

(13) Relation to the Fourier series coefficients of an periodic sequence:

A periodic seq^n $x_p(n)$ with fundamental period N can be represented in a F.S as

$$x_p(n) = \sum_{K=0}^{N-1} C_K e^{j\frac{2\pi}{N}nk} \rightarrow (1) \quad -\infty < n < \infty$$

where the Fourier series coefficients are given by the exp^n

$$C_K = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi}{N}Kn} \rightarrow (2)$$

comparing eq (1) (2) with

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \quad \& \quad x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K) e^{j\frac{2\pi}{N}Kn}$$

$$\boxed{X(K) = N \cdot C_K}$$

1) Find the IDFT of $X(K) = \{1, 0, 1, 0\}$

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, e^{\frac{j2\pi}{N}Kn} \quad ; \quad 0 \le n \le N-1$$

Assume $\boxed{N=4}$

$= \frac{1}{N}$

$$x(n) = \frac{1}{4} \sum_{K=0}^{3} X(K)\, e^{\frac{j2\pi}{4}Kn} \qquad 0 \le n \le 3$$

$$= e^{\frac{j\pi}{2}Kn}$$

$$= \frac{1}{4}\left[X(0) + \underbrace{X(1)}_{=0} e^{\frac{j\pi}{2}n} + X(2)\, e^{j\pi n} + \underbrace{X(3)}_{=0} e^{\frac{j\pi}{2}3n} \right]$$

$$= \frac{1}{4}\left[X(0) + X(2)\, e^{j\pi n} \right]$$

for $n = 0$

$$x(0) = \frac{1}{4}\left[1 + 0 + 1 + 0 \right] = \frac{2}{4} = \underline{\underline{0.5}}$$

$$x(1) = \frac{1}{4}\left[1 + 1 \times e^{j\pi} \right]$$

$e^{j\pi} = \cos\pi + j\sin\pi$
$= (-1) + 0$
$= -1$

$$= \frac{1}{4}\left[1 + 1 \times (-1) \right] = \frac{1}{4}\left[1 - 1 \right] = \underline{\underline{0}}$$

$$x(2) = \frac{1}{4}\left[1 + X(2)\, e^{j2\pi} \right]$$

$e^{j2\pi} = \cos 2\pi + j\sin 2\pi$

$$= \frac{1}{4}\left[1 + 1\, e^{j2\pi} \right] = \frac{1}{4}\left[1 + 1 \right] = \frac{2}{4}$$

$= 1 + 0$

$$= \underline{\underline{0.5}}$$

$= 1$

$$x(3) = \frac{1}{4}\left[1 + 1\, e^{j3\pi} \right] = \frac{1}{4}\left[1 - 1 \right] = \underline{\underline{0}}$$

$e^{j3\pi} = \cos 3\pi + j\sin 3\pi$
$= -1 - 0$

(8) Find the N-point DFT of the seq^n

$$x(n) = \begin{cases} 1 ; & n = \text{even} \\ 0 ; & n = \text{odd} \end{cases} \quad ; \quad 0 \le n \le N-1$$

$$\& \ N = \text{odd}$$

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}Kn} \quad ; \quad 0 \le K \le N-1$$

$$= 1 + x(2) e^{-j\frac{2\pi}{N}K.2} + x(4) e^{-j\frac{2\pi}{N}K.4}$$

$$+ \cdots + x(N-1) e^{-j\frac{2\pi}{N}K(N-1)}$$

$$= \sum_{n=0}^{\frac{N-1}{2}} \left(e^{-j\frac{2\pi}{N}K.2}\right)^m$$

$$\boxed{\text{WKT} \quad \sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1-a}}$$

~~$N_1 = 0 \quad N_2 = e^{-j\frac{2\pi}{N}K.2}$~~ $\therefore$

~~$\Rightarrow N_2 + 1 = e^{-j\frac{4\pi K}{N}}$~~

here $N_1 = 0 \quad N_2 = \frac{N-1}{2} \quad \therefore N_2 + 1 = \frac{N-1}{2} + 1$

$$= \frac{N-1+2}{2}$$

$$= \frac{N+1}{2}$$

$$X(k) = \frac{1 - \left(e^{-j\frac{2\pi}{N}K.2}\right)^{\frac{N+1}{2}}}{1 - e^{-j\frac{4\pi}{N}K}}$$

$=$

$$= \frac{1 - e^{-j\frac{2\pi}{N}k}}{1 - e^{-j\frac{4\pi}{N}k}}$$

$$e^{-j\frac{2\pi}{N}k \cdot 2 \times \frac{N}{2}}$$

$$\cdot\ e^{-j\frac{2\pi}{N}k\frac{2}{2}}$$

$$= e^{-j\cancel{2\pi k}} \cdot e^{-j\frac{2\pi}{N}k}$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$= \frac{1 - e^{-j\frac{2\pi}{N}k}}{\left(1 - e^{-j\frac{2\pi}{N}k}\right)\left(1 + e^{-j\frac{2\pi}{N}k}\right)}$$

(9) Find the 4-point DFT of the sequence $x(n) = [1, 0, 0, 1]$ using matrix method & verify the answer by taking the 4-point IDFT of the result.

(a) $X_N = W_N x_N$

$N = 4$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1+j \\ 0 \\ 1-j \end{bmatrix}$$

$$x_N = \frac{1}{N} W_N^{*} X_N$$

<u>$N=4$</u>

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 2 \\ 1+j \\ 0 \\ 1-j \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 4 \\ 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Properties of DFT

(i) periodicity :

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

then
$$\boxed{\begin{aligned} X(K+N) &= X(K) \quad \forall K \\ x(n+N) &= x(n) \quad \forall n \end{aligned}}$$

The sequences $x(n)$ & $X(K)$ are implicit periodic with a period equal to N

proof: WKT $x(n) \triangleq \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{-Kn}$

replacing n by $(n+N)$ we get

$$x(n+N) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{-K(n+N)}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{-Kn} \cdot \overset{=1}{W_N^{-KN}}$$

WKT $W_N^{-KN} = e^{j\frac{2\pi}{N}KN}$

$= e^{j2\pi K} = 1$ always

$$x(n+N) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{-Kn}$$

$$x(n+N) = x(n) \longrightarrow (1)$$

eq (1) $\Rightarrow$ $x(n)$ is implicit periodic with a period $= N$

$$X(K) \triangleq \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

replace k by K+N, we get

$$X(K+N) = \sum_{n=0}^{N-1} x(n)\, W_N^{(K+N).n}$$

$$= \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \cdot W_N^{Nn}$$

$$W_N^{Nn} = e^{-j\frac{2\pi}{N}Nn} = e^{-j2\pi n}$$

$$= 1 \text{ always}$$

$$X(K+N) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$X(K+N) = X(K) \longrightarrow (2)$$

eq (2) $\Rightarrow$ X(K) is implicit periodic with a period. N.

(1) find a 4-point DFT of the sequence $x(n) = 2^n$, $0 \le n \le 3$. Also find $X(-1)$ & $X(5)$

Soln: given $x(n) = \{\underset{\uparrow}{1}, 2, 4, 8\}$

WKT $W_4^0 = 1$, $W_4^1 = -j$, $W_4^2 = -1$, $W_4^3 = +j$

$$X(K) \triangleq \sum_{n=0}^{3} x(n)\, W_4^{kn}, \quad 0 \le K \le 3$$

$$= 1 + 2W_4^K + 4W_4^{2K} + 8W_4^{3K}$$

$\therefore X(0) = 1+2+4+8 = \boxed{15}$

$X(1) = 1 + 2W_4^1 + 4W_4^2 + 8W_4^3 = 1 - 2j - 4 + 8j = \boxed{-3+6j}$

$X(2) = 1 + 2W_4^2 + 4W_4^4 + 8W_4^6$

$\qquad\qquad\qquad\qquad = 4W_4^0 \quad = 8W_4^2$

$= 1 - 2 + 4 - 8 = \boxed{-5}$

$X(3) = 1 + 2W_4^3 + 4W_4^6 = 4W_4^2 + 8W_4^9 = 8W_4^1$

$= 1 + 2j - 4 - 8j = \boxed{-3-6j}$

Periodic extension of $X(K)$ is

$15, -3+j6, -5, -3-j6 \mid 15, -3+j6, -5, -3-j6 \mid 15, -3+j6, -5, -3-j6$

$K=-1 \mid K=0 \quad 1 \quad 2 \quad 3 \mid 4 \quad 5$

$\therefore X(-1) = -3-j6$

$X(5) = -3+j6$

② Linearity:

The Linearity property of DFT states that:

if $x_1(n) \xleftrightarrow[N]{DFT} X_1(K)$

& $x_2(n) \xleftrightarrow[N]{DFT} X_2(K)$, then

$$a_1x_1(n) + a_2x_2(n) \xleftrightarrow[N]{DFT} a_1X_1(K) + a_2X_2(K)$$

here a_1 & a_2 are constants.

Proof: By definition of DFT,

$$X(K) \triangleq \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \rightarrow (1)$$

Let $x(n) = a_1x_1(n) + a_2x_2(n)$

then eq (1) becomes

$$X(K) = \sum_{n=0}^{N-1} [a_1 x_1(n) + a_2 x_2(n)] W_N^{Kn}$$

$$= \sum_{n=0}^{N-1} a_1 x_1(n) W_N^{Kn} + \sum_{n=0}^{N-1} a_2 x_2(n) W_N^{Kn}$$

$$= a_1 \sum_{n=0}^{N-1} x_1(n) W_N^{Kn} + a_2 \sum_{n=0}^{N-1} x_2(n) W_N^{Kn}$$

$$= a_1 X_1(K) + a_2 X_2(K)$$

(2) compute 4-point DFT of the seqn $x(n)$ given below using linearity property.
(VTU)

$$x(n) = \cos(\tfrac{\pi}{4} n) + j \sin(\tfrac{\pi}{4} n) \quad , \quad 0 \le n \le 3$$

Soln:-

Let $x(n) = x_1(n) + j x_2(n)$

where $x_1(n) = \cos(\frac{\pi}{4} n)$ & $x_2(n) = \sin(\frac{\pi}{4} n)$

n	$\cos(\frac{\pi}{4} n)$	$\sin(\frac{\pi}{4} n)$
0	1	0
1	$1/\sqrt{2}$	$1/\sqrt{2}$
2	0	1
3	$-1/\sqrt{2}$	$1/\sqrt{2}$

To find $X_1(K)$ & $X_2(K)$

$W_4^0 = 1, \quad W_4^1 = -j, \quad W_4^2 = -1, \quad W_4^3 = +j$

$$X_1(K) = \sum_{n=0}^{3} x_1(n) W_4^{Kn}$$

$$= 1 + \frac{1}{\sqrt{2}} W_4^{K} - \frac{1}{\sqrt{2}} W_4^{3K}$$

$X_1(0) = 1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \boxed{1}$, $X_1(1) = 1 + \frac{1}{\sqrt{2}} W_4^1 - \frac{1}{\sqrt{2}} W_4^3$

$$= 1 - \frac{1}{2} j - \frac{1}{\sqrt{2}} j$$

$$\boxed{= 1 - 1.414 j}$$

$\cancel{X_2(}$

(or)

using matrix method

$\cancel{X_1(0)}$

$\cancel{X_2(}$

$$X_{1N} = [W_N] \cdot x_N$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0.707 \\ 0 \\ -0.707 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 - j1.414 \\ 1 \\ 1 + j1.414 \end{bmatrix}$$

IIIy

$$X_2(K) = \sum_{n=0}^{3} x_2(n) W_4^{Kn} , \quad 0 \leq K \leq 3$$

$$X_{2N} = [W_N] \cdot x_N$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 0.707 \\ 1 \\ 0.707 \end{bmatrix}$$

$$= \begin{bmatrix} 2.414 \\ -j \\ +j0.414 \\ -1 \end{bmatrix}$$

$$X(K) = X_1(K) + j X_2(K) \qquad K = 0, 1, 2, 3$$

$$= (1 + j2.414) + (1 - j1.414 - 1j)$$
$$+ (1 - j0.414) + (1 + j1.414 - j1)$$

4 16

circular time shift (or) circular Translation

WKT, N-point DFT of a finite durⁿ sequence $x(n)$ of length $L \le N$ is equivalent to N-point DFT of a periodic seqⁿ $x_p(n)$ of period N, which is obtained by periodically extending $x(n)$.

$$x_p(n) = \sum_{l=-\infty}^{+\infty} x(n - lN) \longrightarrow (1)$$

i.e,

$$x(n) \xleftrightarrow[N]{DFT} X(K) \longrightarrow (2)$$

$$x_p(n) \xleftrightarrow[N]{DFT} X(K) \longrightarrow (3)$$

eq (2) & (3) are related by eq (1).
& can be rewritten as

$$x(n) = \begin{cases} x_p(n), & 0 \le n \le N-1 \\ 0, & E.W \end{cases} \longrightarrow (4)$$

* Let $x_p(n)$ be shifted by 'K' units to right then the new seqⁿ $x'_p(n)$ is

$$x'_p(n) = x_p(n-K) \longrightarrow (5)$$

$$= \sum_{l=-\infty}^{\infty} x(n-K-lN) \longrightarrow (6)$$

Then the corresponding seqⁿ $x'(n)$ can be obtained from eq (4)

$$x'(n) = \begin{cases} x'_p(n) & 0 \le n \le N-1 \\ 0, & e.w \end{cases} \longrightarrow (7)$$

The seqⁿ $x'(n)$ is related to $x(n)$ by the circular shift. This concept is illustrated in fig below.

Let

$x(n) \rightarrow$ contains 4-samples i.e $N=4$ & is shown in fig (1a).

$x_p(n) \rightarrow$ periodic repetition of $x(n)$ is shown in fig (1b)

$x'_p(n) \rightarrow$ which is obtained by shifting $x_p(n)$ to right by 2 samples is shown in fig (1c)

$x'_p(n) = x_p(n-2)$, $x'(n) \rightarrow$ shown in fig (1d) as per eq (7)

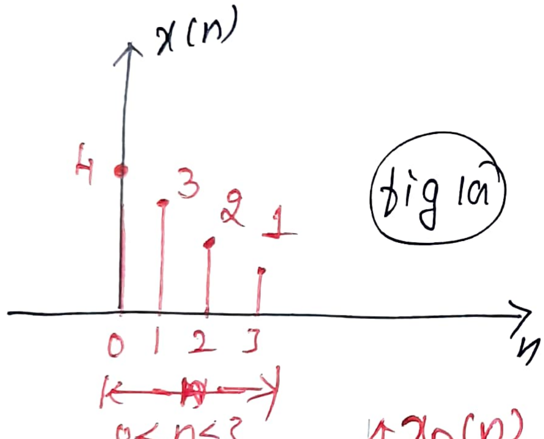


fig 1a

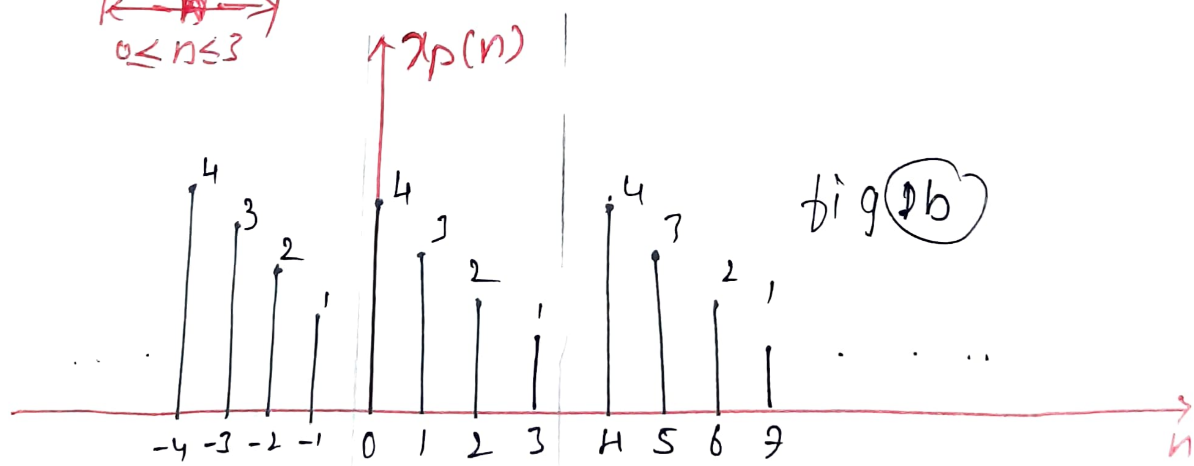


fig (1b)

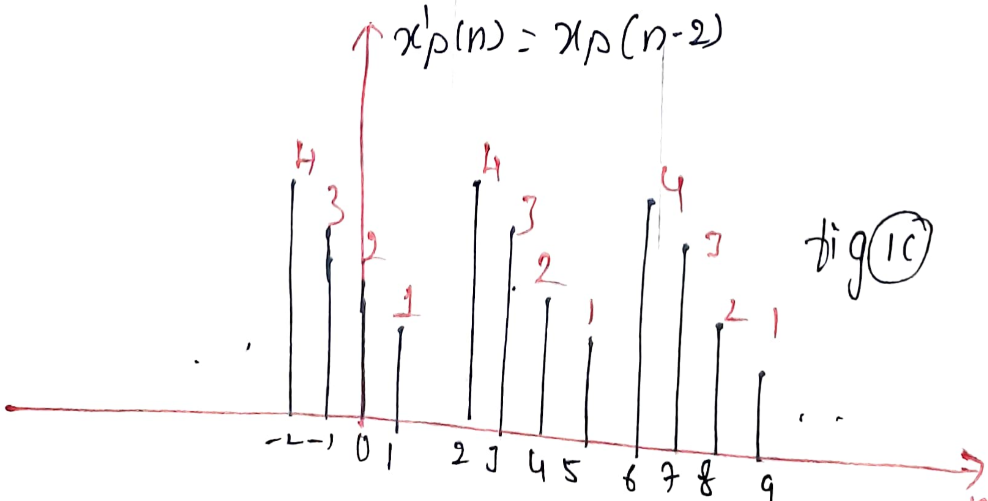


fig (1c)

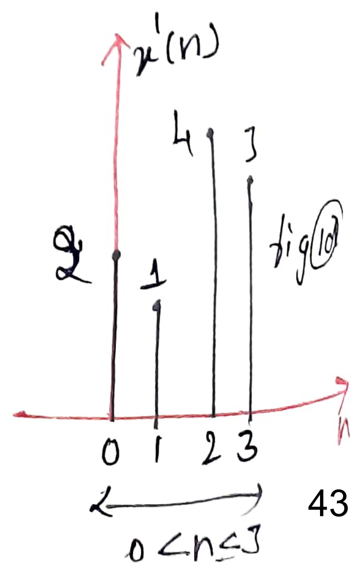


fig (1d)

The seq^n $x'(n)$ is related to $x(n)$ by a circular shift & is represented as

$$\boxed{\begin{aligned} x'(n) &= x(n-k,\ \text{modulo } N) \\ &= x((n-k))_N \end{aligned}} \longrightarrow (8)$$

Let us evaluate $x'(n)$ as per eq (8) with $N = 4$ $k = 2$

$x'(n) = x((n-2))_4 \quad = x((n-k))_N$

$\therefore x'(0) = x((0-2))_4 = x((-2))_4 = x(2) = 2$

[$x(N-2) = x(4-2) = x(2)$]

$x'(1) = x((1-2))_4 = x((-1))_4 = x(3) = 1$

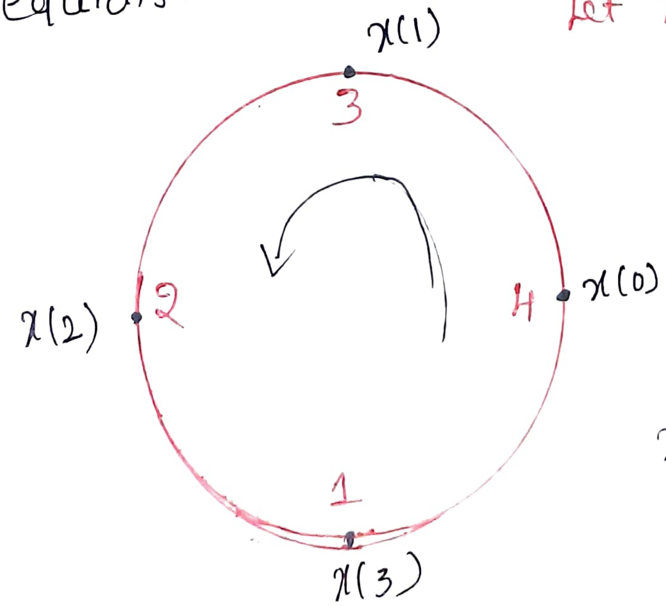
[$x(N-1) = x(4-1) = x(3)$]

$x'(2) = x((2-2))_4 = x(0) = 4$

$x'(3) = x((3-2))_4 = x(1) = 3$

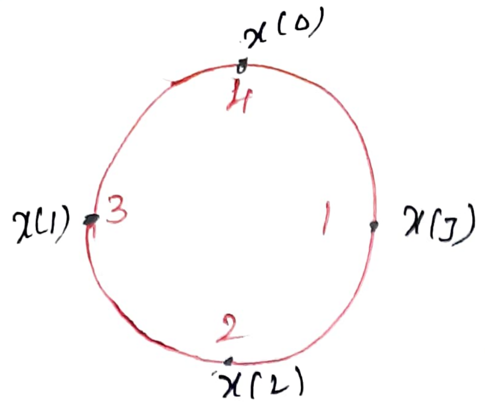
* Another simple way is to write the samples of seq^n on the circumference of a circle at equidistant in anticlockwise dir^n.

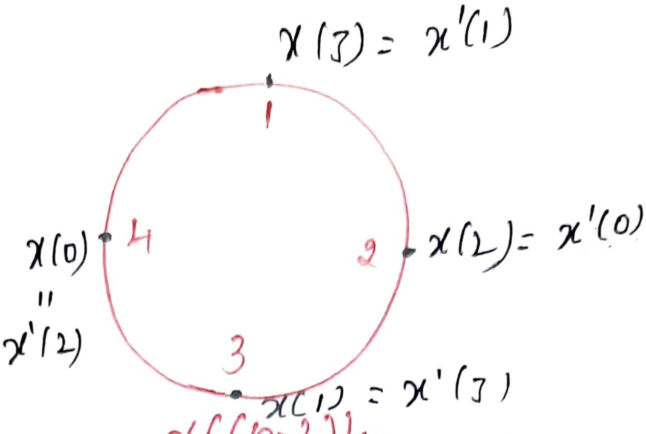
Let $x(n) = \{4, 3, 2, 1\}$



$x(n)$ is written in anticlockwise dir^n.

$x((n))_4$





$x((n-1))_4$

(b) seq $x(n)$ shifted by one sample in anticlock wise dirn

$x((n-2))_4$

(c) seq $x(n)$ shifted by 2 samples in anticlockwise dirn

$\boxed{x'(n) : \{2, 1, 4, 3\}}$

<u>Note</u> :-

$x((n-k))_N$, if k is +ve, the circle is to be rotated in anticlockwise dirn by 'k' steps [~~+ve~~ [right circular shift]

if 'k' is -ve, the circle is to be rotated in clockwise dirn by 'k' steps [left shift]

<u>circularly even sequence</u>

A seq^n is said to be circularly even, if it is symmetric about the point zero on the circle

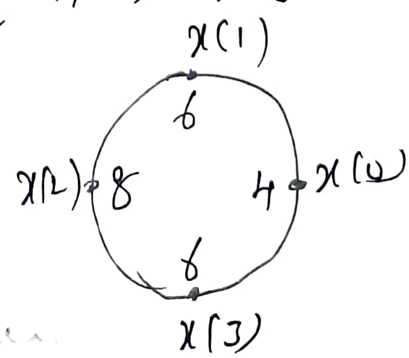
is, $\boxed{x(N-n) = x(n) \quad 1 \le n \le N-1} \rightarrow (1)$

consider the seq $x(n) : \{4, 6, 8, 6\}$

$x(4-1) = x(3) = x(1)$

$x(4-2) = x(2) = x(2)$

$x(4-3) = x(1) = x(3)$



no need

circularly odd sequence

A seq^n is said to be circularly odd, if it is antisymmetrical about point $x(0)$ on the circle ie,

$$\boxed{x(N-n) = -x(n), \quad 1 \le n \le N-1} \rightarrow (2)$$

consider the seq^n $x(n) = \{+4, -6, 8, 6\}$

$$\left\{\begin{array}{l} x(4-1) = x(3) = -x(1) \\ x(4-2), \; x(2) = -x(2) \\ x(4-3), \; x(1) = -x(3) \end{array}\right\}$$

no need to explain this

circularly folded seq^n:

A circularly folded seq^n is represented as $x((-n))_N$. It is obtained by ploting $x(n)$ in clockwise dir^n along the circle & is represented as

$$x((-n))_N = x(N-n), \quad 0 \le n \le N-1$$

(1) A length-6 seq^n is given by

$$x(n) = \{1, 3, 2, 5, -2, 7\}$$

obtain $y(n) = x((n-4))_6$

soln:

$$x(n) = \{\underset{0}{1}, \underset{1}{3}, \underset{2}{2}, \underset{3}{5}, \underset{4}{-2}, \underset{5}{7}\}$$

$y(n) = x((n-4))_6$

$y(0) = x((0-4))_6 = x((-4))_6 = x(6-4) = x(2) = 2$

$y(1) = x((1-4))_6 = x((-3))_6 = x(6-3) = x(3) = 5$

$y(2) = x($

$\vdots$

$y(n) = \{2, 5, -2, 7, 1, 3\}$

(2) $x(n) = \{0, 3, 4, -1, 4, 2, 8, 9, 2, 3\}$

find $x'(n) = x((n-7))_{10}$

$y(n) = x((n+3))_{10}$

$\{-1, 4, 2, 8, 9, 2, 3, 0, 3, 4\}$

(3) $x(n) = \{6, 5, 4, 3\}$. sketch $x'(n) = x((n-2))_4$

$x(n) : \{6, 5, 4, 3\}$

$x'(n-1) : \{3\ 6\ 5\ 4\}$

$x(n-2) : \{4\ 3\ 6\ 5\}$

$x_1(n) : x((n-2))_4 = \{4, 3, 6, 5\}$

(3) circular - time shift property :

(UK)

if $x(n) \xleftrightarrow[N]{DFT} X(K)$

then $x((n-l))_N \xleftrightarrow[N]{DFT} X(K)\, e^{-j\frac{2\pi}{N}Kl}$

(or)

$X(K)\, W_N^{Kl}$

Proof :

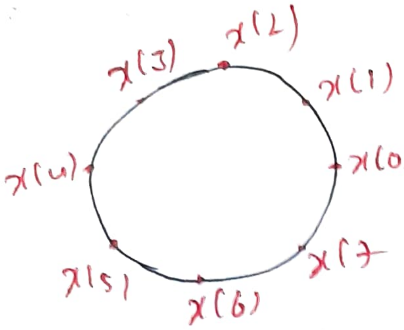
from the defination of DFT,

$$DFT\{x((n-l))_N\} = \sum_{n=0}^{N-1} x((n-l))_N\, e^{-j\frac{2\pi}{N}kn}$$

we can split the summations in to 2 parts

$$= \underbrace{\sum_{n=0}^{l-1} x((n-l))_N\, e^{-j\frac{2\pi}{N}kn}}_{I} + \underbrace{\sum_{n=l}^{N-1} x(n-l)\, e^{-j\frac{2\pi}{N}kn}}_{II} \rightarrow (1)$$

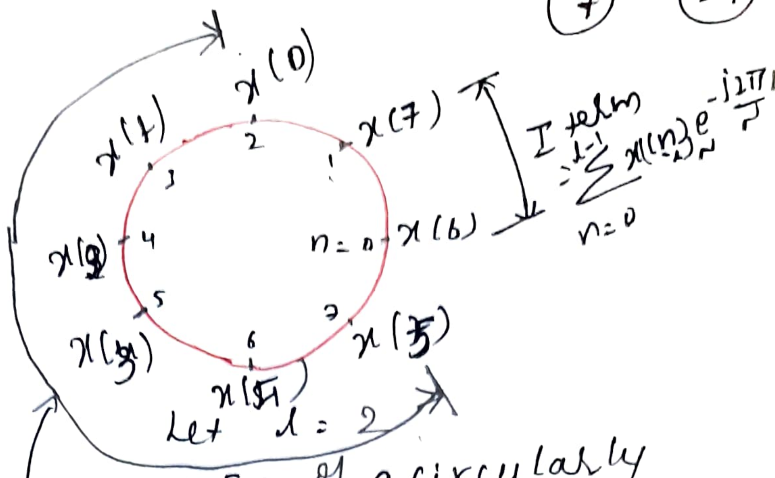
(for $n = 0$ to $l-1$, the seqn is circularshifted, remaining values of l to $N-1$ - linear shifiting) Ref fig below



A seq^n having 8 samples plotted on a ck

$x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7$

$x_7\ x_0\ x_1\ x_2\ x_3\ x_4\ x_5 \}$



DFT of a circularly shifted seq^n

II term $\sum_{n=l}^{N-1} x(n-l) e^{-j\frac{2\pi}{N}kn}$

but $x((n-l))_N = x(N-l+n)$ since there is circular shift.

consider I term

$$\sum_{n=0}^{l-1} x((n-l))_N \, e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{l-1} x(N-l+n) \, e^{-j\frac{2\pi}{N}kn}$$

put $m = N-l+n$ in the above eq^n.

$$= \sum_{m=N-l}^{N-1} x(m) \, e^{-j\frac{2\pi}{N}k(m+l-N)}$$

$$= \sum_{m=N-l}^{N-1} x(m) \, e^{-j\frac{2\pi}{N}k(m+l)} \, \cancel{e^{j\frac{2\pi}{N}kN}}^{=1}$$

$$= \sum_{m=N-l}^{N-1} x(m) \, e^{-j\frac{2\pi}{N}k(m+l)} \longrightarrow (2)$$

II Term $\rightarrow \sum_{n=l}^{N-1} x(n-l) e^{-j\frac{2\pi}{N}kn}$

put $m = n-l$

$$= \sum_{m=0}^{N-l-1} x(m) e^{-j\frac{2\pi}{N}k(m+l)} \longrightarrow (3)$$

$\therefore$

$$DFT\{x((n-l))_N\} = \text{II term} + \text{I term}$$

$$= \sum_{m=0}^{N-l-1} x(m) e^{-j\frac{2\pi}{N}k(m+l)}$$

$$+ \sum_{m=N-l}^{N-1} x(m) e^{-j\frac{2\pi}{N}k(m+l)}$$

The above summations can be combined into single one. Note that even though we have assumed 2 diff' values for m it is just an index. first summation in above eqn is performed from 0 to $N-l-1$ & 2nd summation is from $N-l$ to $N-1$. $\therefore$ The over all summation can be replaced by 0 to $N-1$.

$$DFT\{x((n-l))_N\} = \sum_{m=0}^{N-1} x(m) e^{-j\frac{2\pi}{N}km} e^{-j\frac{2\pi}{N}kl}$$

$$= X(K) e^{-j\frac{2\pi}{N}kl}$$

$$= X(K) W_N^{Kl}$$

hence the proof

① Find 4-point DFT of the seq^n

$x(n) = \delta(n) + 2\delta(n-1) + 3\delta(n-2) + 4\delta(n-3)$

Also find the 4-point DFT $Y(K)$ if

$y(n) = x((n-2))_4$

Soln:

$$DFT\{x(n)\} = X(K) \triangleq \sum_{n=0}^{N-1} x(n) W_N^{Kn}$$

$$X(K) = \sum_{n=0}^{3} [\delta(n) + 2\delta(n-1) + 3\delta(n-2) + 4\delta(n-3)] W_4^{Kn}$$

$0 \leq K \leq 3$.

Applying shifting property, we get.

$$X(K) = W_4^{Kn}\Big|_{n=0} + 2W_4^{Kn}\Big|_{n=1} + 3W_4^{Kn}\Big|_{n=2} + 4W_4^{Kn}\Big|_{n=3}$$

$$= 1 + 2W_4^{K} + 3W_4^{2K} + 4W_4^{3K}$$

$W_4^0 = 1, \quad W_4^1 = -j, \quad W_4^2 = -1, \quad W_4^3 = +j$

$X(K) = \{10, \ -2+2j, \ -2, \ -2-j2\}$

using time-shifting property

$$DFT\{x((n-l))_N\} \xleftrightarrow[N]{DFT} X(K) W_N^{Kl}$$

$$y(n) = x((n-2))_4 \xleftrightarrow[N]{DFT} X(K) W_4^{2K}.$$

$(l = 2, \ N = 4)$

$y(0) = X(0) W_4^{2.0} = 1 \times 10 = \underline{\underline{10}}$

$y(1) = X(1) W_4^{2} = (-2+j2)(-1) = \underline{\underline{2-j2}}$

$y(2) = X(2) W_4^{4} = X(2) W_4^{0}$

$= (-2)(1) = -2$

$y(3) = W_4^{6} \cdot X(3) = (-1)(-2-j2) = 2+j2$

$\quad\ \ \| \ W_4^2$

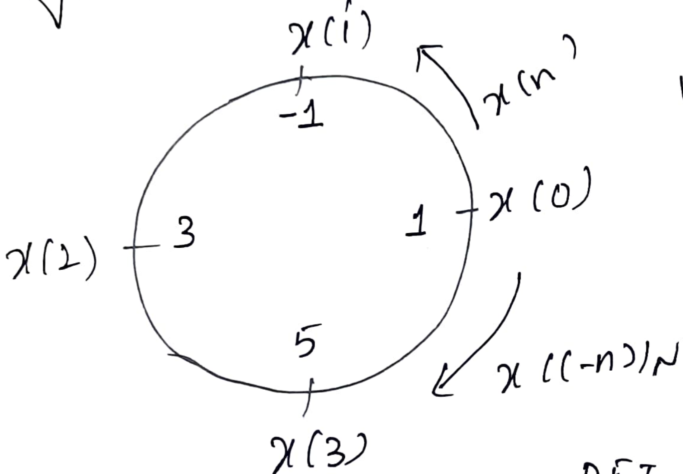
$Y(K) = \{10, \ 2-j2, \ -2, \ 2+j2\}$

(H) Time Reversal property

(U14) WKT $x(n)$ is written on the circle at equidistance in anticlockwise diren. If the same seqn $x(n)$ is written in clockwise diren. Then we will get the time Reversal of $x(n)$ which is denoted by

$x((-n))_N$

eg $x(n) = \{\underset{\uparrow}{1}, -1, 3, 5\}$



Now
$y(n) = x((-n))_4$

$y(n) = \{\underset{\uparrow}{1}, 5, 3, -1\}$

Statement: if $x(n) \xleftrightarrow[N]{DFT} X(K)$

Then

$y(n) = x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT} Y(K)$

$= X((-K))_N$

$= X(N-K)$

[DFT is periodic over period 'N'. $X(N-K)$ is equivalent to folding $X(K)$]

Proof:

$Y(K) = DFT\{x((-n))_N\} = DFT\{x(N-n)\}$

$$DFT\{x(N-n)\} \triangleq \sum_{n=0}^{N-1} x(N-n)\, e^{-j\frac{2\pi}{N}kn} \longrightarrow (1)$$

put $m = N-n$ then $n = N-m$

$$= \sum_{m=N}^{1} x(m)\, e^{-j\frac{2\pi}{N}K(N-m)} \longrightarrow (2)$$

$$= \sum_{m=1}^{N} x(m)\, e^{j\frac{2\pi}{N}km} \cdot e^{-j\frac{2\pi}{N}kN}$$

$$\because \quad e^{-j\frac{2\pi}{N}kN} = e^{-j2\pi k} = 1 \text{ always} \longrightarrow (3)$$

$$= \sum_{m=1}^{N} x(m)\, e^{j\frac{2\pi}{N}km}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{j\frac{2\pi}{N}km} \longrightarrow (4)$$

On the basis of eq (3) we can write

$$e^{-j2\pi m} = \cos(2\pi m) - j\sin(2\pi m)$$
$$= 1 \quad \forall m$$

hence if we multiply RHS of eq (4) by $e^{-j2\pi m}$ its meaning will n't change

$$DFT\{x(N-n)\} = \sum_{m=0}^{N-1} x(m)\, e^{j\frac{2\pi}{N}km} \cdot e^{-j2\pi m}$$

Let us rearrange $e^{-j2\pi m}$ as $e^{-j\frac{2\pi}{N}mN}$.

Then,

$$= \sum_{m=0}^{N-1} x(m)\, e^{j\frac{2\pi}{N}km} \cdot e^{-j\frac{2\pi}{N}mN}$$

$$= \sum_{m=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}m(N-k)}$$

by defination of DFT, the R.H.S of the above eq 4 is

$$DFT\{x(N-n)\} = X(N-k)$$ hence the proof

$$= X((-k))_N$$

(11) Obtain the DFT of the foll seqn $x_1(n) = \{1, 1, 2, 3\}$ & also find DFT of $x_2(n) = x_1((-n))_4$.

$x_1(n) = \{1, 1, 2, 3\}$

$X_1(k) = \{7, -1+j2, -1, -1-j2\}$

~~$X_2(k)$~~

WKT $x((-n))_N \xleftrightarrow[N]{DFT} X((-k))_N$

$X_2(k) = DFT\{x_1((-n))_4\} = X_1((-k))_4$

$X_2(0) = X_1((-0))_4 = X_1(0) = 7$

$X_2(1) = X_1((-1))_4 = X_1(4-1)$
$= X_1(3) = -1-j2$

$X_2(2) = X_1((-2))_4 = X_1(4-2)$
$= X_1(2) = -1$

$X_2(3) = X_1((-3))_4 = X_1(4-3)$
$= X_1(1) = -1+j2$

$\therefore$

$X_2(k) = \{7, -1-j2, -1, -1+j2\}$

(10) (28)

(5) circular frequency shift :

if $x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)$

then $x(n)\, e^{j\frac{2\pi}{N} l n} \overset{DFT}{\underset{N}{\longleftrightarrow}} X((k-l))_N$

proof :-

$$y(n) = IDFT\{ X((k-l))_N \}$$

$$= \frac{1}{N}\left[\sum_{k=0}^{N-1} X((k-l))_N \, e^{j\frac{2\pi}{N} kn} \right] \longrightarrow (1)$$

$$= \frac{1}{N}\left[\underbrace{\sum_{k=0}^{l-1} X((k-l))_N \, e^{j\frac{2\pi}{N} kn}}_{I - term} + \underbrace{\sum_{k=l}^{N-1} X(k-l)\, e^{j\frac{2\pi}{N} kn}}_{II - term} \right]$$

II - term $\longrightarrow$ (2)

I - term

$$\sum_{k=0}^{l-1} X((k-l))_N \, e^{j\frac{2\pi}{N} kn}$$

$$= \sum_{k=0}^{l-1} X(N-l+k)\, e^{j\frac{2\pi}{N} kn}$$

put $m = N-l+k$

$$= \sum_{m=N-l}^{N-1} x(m)\, e^{j\frac{2\pi}{N} n [m+l-N]}$$

$$= \sum_{m=N-l}^{N-1} x(m)\, e^{j\frac{2\pi}{N} n(m+l)} \cdot \underbrace{e^{-j\frac{2\pi}{N} n \cdot N}}_{=1}$$

$$= \sum_{m=N-l}^{N-1} x(m)\, e^{j\frac{2\pi}{N} n(m+l)} \longrightarrow (3)$$

II - term :

$$\sum_{K=l}^{N-1} X(K-l)\, e^{j\frac{2\pi}{N}kn}$$

put $m = k - l$ $\therefore$ $k = m + l$

$$= \sum_{m=0}^{N-l-1} x(m)\, e^{j\frac{2\pi}{N}n(m+l)} \longrightarrow (H)$$

$\therefore y(n) = IDFT\{X((K-l))_N\} = \frac{1}{N}[\text{II term} + \text{I term}]$

$$= \frac{1}{N}\left[\sum_{m=0}^{N-l-1} x(m)\, e^{j\frac{2\pi}{N}n(m+l)} + \sum_{m=N-l}^{N-1} x(m)\, e^{j\frac{2\pi}{N}n(m+l)}\right]$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x(m)\, e^{j\frac{2\pi}{N}n(m+l)}$$

$$= \underbrace{\frac{1}{N}\sum_{m=0}^{N-1} x(m)\, e^{j\frac{2\pi}{N}nm}} \cdot e^{j\frac{2\pi}{N}nl}$$

$$X((K-l))_N = x(n) \cdot e^{j\frac{2\pi}{N}nl}$$

$$= x(n)\, W_N^{-nl}$$ hence

the proof

① find 4-point DFT of the seqn given below. $x(n)=\{1,-1,-1,1\}$. Also find $y(n)$ if $Y(K)=X((K-2))_4$

Soln

$$DFT\{x(n)\}=X(K)=\sum_{n=0}^{3} x(n)W_4^{kn}$$

$$X(K)=1-W_4^{K}-W_4^{2k}+W_4^{3k}$$

$$X(K)=\{0,\ 2+2j,\ 0,\ 2-j2\}$$

Recalling the shifting property

$$DFT\{W_N^{-ln}x(n)\}=X((K-l))_N$$

$$IDFT\{X((K-l))_N\}=x(n).W_N^{-ln}$$

$$\therefore\ Y(K)=X((K-2))_4$$

$$\therefore\ y(n)=W_4^{-2n}x(n)$$

$$y(0)=W_4^{-0}x(0)=[W_4^0]^*x(0)=1*1=\underline{\underline{1}}$$

$$y(1)=W_4^{-2}x(1)=[W_4^2]^*x(1)=-1*-1=1$$

$$y(2)=W_4^{-0}x(2)=[W_4^0]^*x(1)=1*-1=-1$$

$$y(3)=W_4^{-2}x(3)=[W_4^2]^*x(3)=-1\times1=-1$$

$$y(n)=\{1,\ 1,\ -1,\ -1\}$$

(6) circular convolution property

Before studying the property, let us discuss the concept of circular-convolution.

Let $x_1(n)$ & $x_2(n)$ be two N-length seqn's. The circular convolution of these sequence is defined as

$$y(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N \longrightarrow (1)$$

$$; 0 \le n \le N-1$$

This operation involves 2-length 'N' seqn's & so it is also referred as N-point circular convolution & is denoted as

$$x_1(n) \, Ⓝ \, x_2(n) = x_2(n) \, Ⓝ \, x_1(n) \longrightarrow (2)$$

(1) $x_1(n) = \{1, 2, 0, 1\}$, $x_2(n) = \{2, 2, 1, 1\}$

find $y(n) = x_1(n) \, Ⓝ \, x_2(n)$.

The 4-point circular convolution of these 2 seq^ns is given by (12)

$$y(n) = x_1(n) \textcircled{N} x_2(n)$$

$$= x_1(n) \textcircled{4} x_2(n) \; ; \quad 0 \le n \le 3$$

$$= \sum_{m=0}^{3} x_1(m)\, x_2((n-m))_4$$

$\underline{n=0}$

$$y(0) = \sum_{m=0}^{3} x_1(m)\, x_2((-m))_4$$

$$= x_1(0)\, x_2((-0))_4 + x_1(1)\, x_2((-1))_4 + x_1(2)\, x_2((-2))_4 + x_1(3)\, x_2((-3))_4$$

$$= x_1(0) x_2(0) + x_1(1) x_2(3) + x_1(2) x_2(2) + x_1(3) x_2(1)$$

$$= 1\cdot 2 + 2\cdot 1 + 0\cdot 1 + 1\cdot 2$$

$$= \underline{\underline{6}}$$

$\underline{n=1}$

$$y(1) = \sum_{m=0}^{3} x_1(m)\, x_2((1-m))_4$$

$$= x_1(0)\, x_2((1-0))_4 + x_1(1)\, x_2((1-1))_4 + x_1(2)\, x_2((1-2))_4 + x_1(3)\, x_2((1-3))_4$$

$$= x_1(0)\, x_2(1) + x_1(1)\, x_2(0) + x_1(2)\, x_2(3) + x_1(3)\, x_2(2)$$

$$= 1\cdot 2 + 2\cdot 2 + 0\cdot 1 + 1\cdot 1$$

$$= \underline{\underline{7}}$$

$\underline{n=2}$

$$y(2) = \sum_{m=0}^{3} x_1(m)\, x_2((2-m))_4$$

$$= x_1(0) x_2(2) + x_1(1) x_2(1) + x_1(2) x_2(0) + x_1(3) x_2(1)$$

$$= \underline{\underline{6}}$$

$n=3$

$$y(3) = \sum_{m=0}^{3} x_1(m)\, x_2((3-m))_4 = x_1(0)\, x_2(3) + x_1(1)\, x_2(2) + x_1(2)\, x_2(1) + x_1(3)\, x_2(0)$$

$$= 5$$

$$y(n) = \{6, 7, 6, 5\}.$$

mainly there are 2 methods to find the circular convolution of 2 length-N seq (13)

ⓐ Concentric ⊙le method:

given 2 length-N seqn's $x_1(n)$ & $x_2(n)$, the N-point circular convolution of these 2 seqn $y(n) = x_1(n) \; (N) \; x_2(n)$ can be found by performing the foll steps.

(i) write 'N' samples of $x_1(n)$ at equidistant around an outer ⊙le in anticlockwise dirn

(ii) Starting at the same point as $x_1(n)$ ⊙le, write N samples of $x_2(n)$ at equidistant around the inner ⊙le in clockwise dirn

(iii) multiply corresponding samples on the 2 ⊙ls & sum the products to get the o/p.

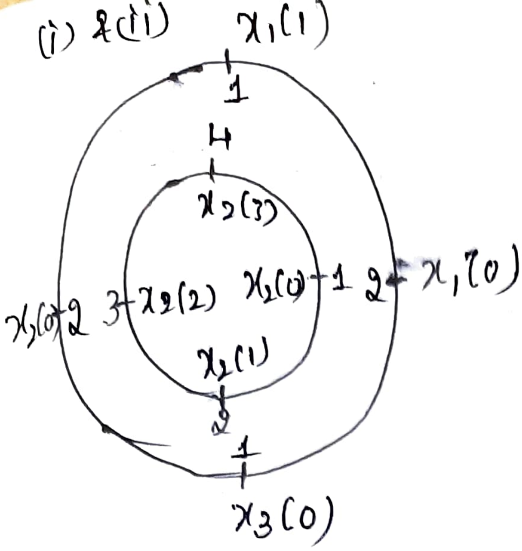
(iv) Rotate the inner ⊙le by one sample in anticlockwise dirn & repeate step (iii) to obtain the next samples of the o/p

(v) Repeat step (iv) for N-1 times to get all the samples of the o/p $y(n)$.

eg. $x_1(n) = \{2, 1, 2, 1\}$

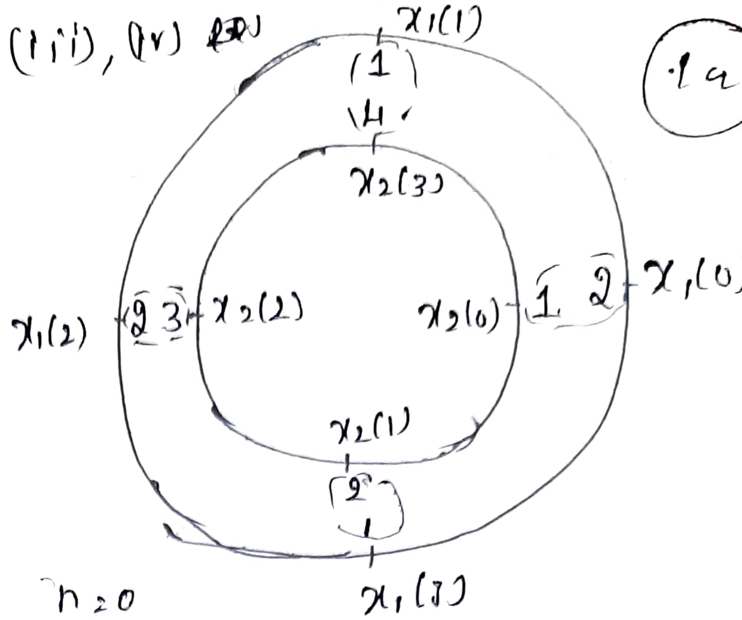
& $x_2(n) = \{1, 2, 3, 4\}$

(i) 2(ii)



(iii), (iv)

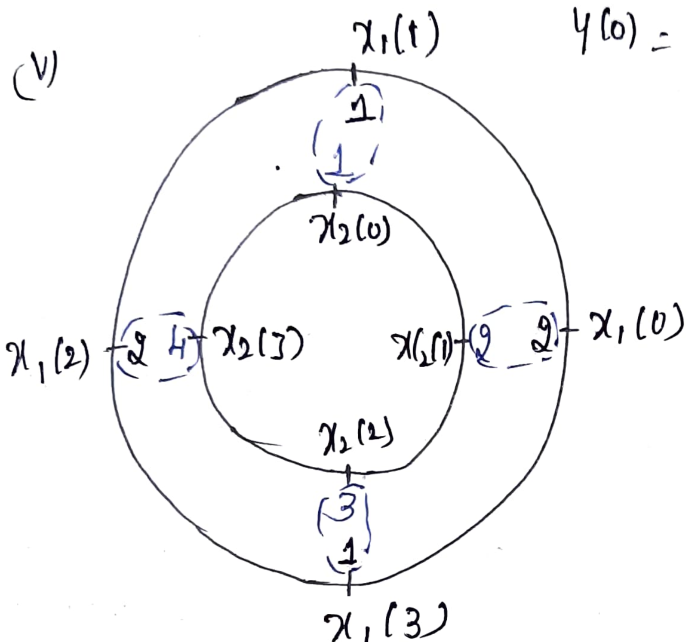
(1a)

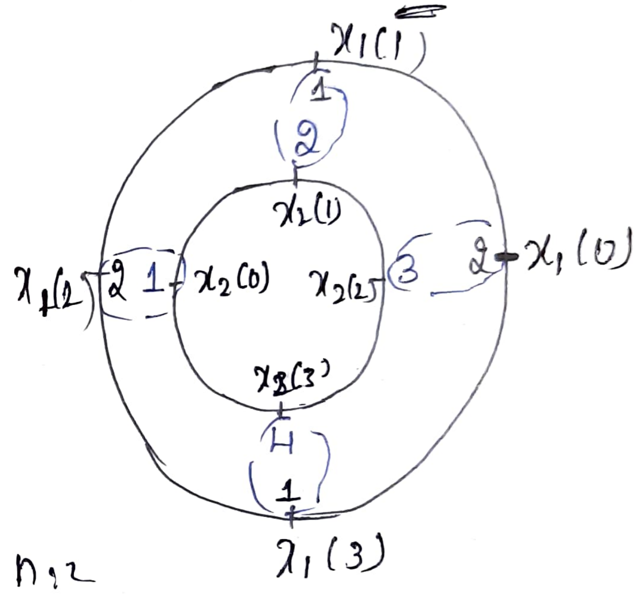


$n=0$

$y(0) = 2+4+6+2 = 14$

(v)



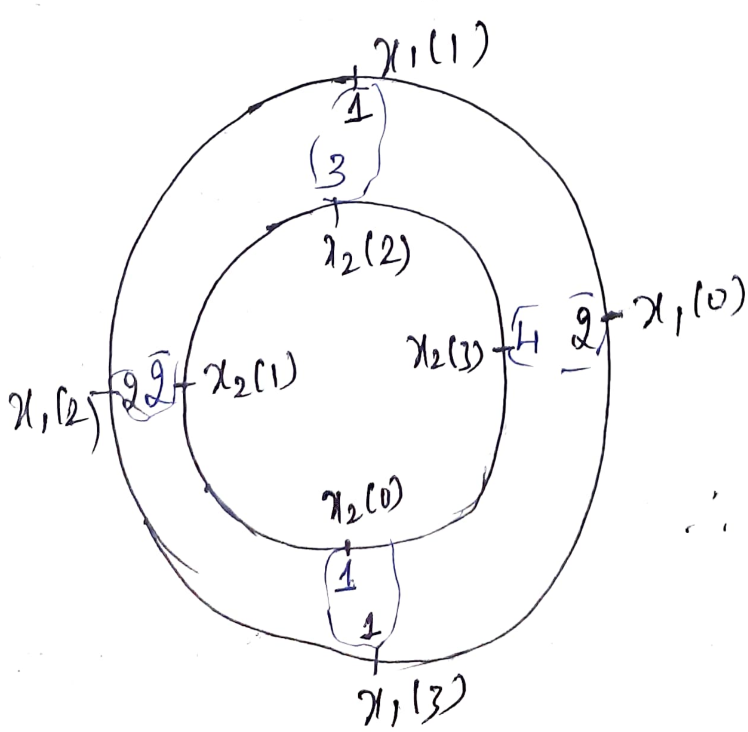


$n=1$

$y(1) = 4+1+8+3 = 16$

$n=2$

$y(2) = 6+2+2+4 = 14$



$n=3$

$y(3) = 8+3+4+1$
$= 16$

$\therefore y(n) = \{14, 16, 14, 16\}$

(1) Compute the circular convolution of the 2 seqn's given below $x(n) = \{1, 4, 2, 6\}$

$h(n) = \{1, 2, 3, 4\}$

Let $y_c(n) = x(n) \, (N) \, h(n) = \sum_{m=0}^{3} x(m) \, h((n-m))_4$

(a) circul.

n	$x(m)$	$h((n-m))_4$	$y(n)$
0	$\{1,4,2,6\}$	$\{1,4,3,2\}$	$1+16+6+12 = 35$
1	$\{1,4,2,6\}$	$\{2,1,4,3\}$	$2+4+8+18 = 32$
2	$\{1,4,2,6\}$	$\{3,2,1,4\}$	$3+8+2+24 = 37$
3	$\{1,4,2,6\}$	$\{4,3,2,1\}$	$4+12+4+6 = 26$

$y_c(n) = \{35, 32, 37, 26\}$

(b) linear conv

$x(n) = \{1,4,2,6\}$ & $h(n) = \{1,2,3,4\}$

$y_L(n) = x(n) * h(n)$

$= [\delta(n) + 4\delta[n-1] + 2\delta[n-2] + 6\delta[n-3]]$

$* \; [\delta(n) + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3]]$

$= \delta(n) + 6\delta(n-1) + 13\delta(n-2) + 26\delta(n-3)$

$+ 34\delta(n-4) + 26\delta(n-5) + 24\delta(n-6)$

$y_L(n) = \{1, 6, 13, 26, 34, 26, 24\}$

$= \{1+34, 6+26, 13+24, 26\} = \{35, 32, 37, 26\}$

$\therefore$ Circular conv = linear conv + Aliasing

(b) **matrix multiplication method**

In this method, the N-point circular convolution of 2 length-N seqns $x_1(n)$ & $x_2(n)$ can be obtained as below. (16)

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(3) \end{bmatrix} = \begin{bmatrix} x_1(0) & x_1(N-1) & x(N-2) & \cdots & x_1(1) \\ x_1(1) & x_1(0) & x_1(N-1) & \cdots & x_1(2) \\ x_1(2) & x_1(1) & x(0) & \cdots & x_1(3) \\ \vdots & \vdots & \vdots & & \\ x_1(N-1) & x_1(N-2) & x_1(N-3) & \cdots & x_1(0) \end{bmatrix} \begin{bmatrix} x_2(0) \\ x_2(1) \\ x_2(2) \\ \vdots \\ x_2(N-1) \end{bmatrix}$$

The seqn $x_1(n)$ is written in each column repeatedly by circularly shifting the samples of it in a NXN matrix.

& such a matrix is called circulant matrix

If $x_1(n) = \{2,1,2,1\}$ $x_2(n) = \{1,2,3,4\}$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2.1 + 1.2 + 2.3 + 1.4 \\ 1.1 + 2.2 + 1.3 + 2.4 \\ 2.1 + 1.2 + 2.3 + 1.4 \\ 1.1 + 2.2 + 1.3 + 2.4 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \\ 14 \\ 16 \end{bmatrix}$$

$\therefore$ $y(n) = x_1(n) \, (4) \, x_2(n)$

$= \{14, 16, 14, 16\}$

Circular convolution Property

(17)

(U.K)

if $x_1(n) \xleftrightarrow[N]{DFT} X_1(K)$.

& $x_2(n) \xleftrightarrow[N]{DFT} X_2(K)$

Then $x_1(n) \; (N) \; x_2(n) \xleftrightarrow[N]{DFT} X_1(K) \cdot X_2(K)$

multiplication of 2 DFTs is equivalent to circular convolution of their seqns in time domain

Proof:

By defn.

$$X_1(K) \triangleq \sum_{n=0}^{N-1} x_1(n) \, e^{\frac{-j2\pi Kn}{N}} \quad , K = 0, \ldots N-1 \longrightarrow (1)$$

$$\& \quad X_2(K) \triangleq \sum_{l=0}^{N-1} x_2(l) \, e^{-j\frac{2\pi}{N} Kl} \quad , K = 0, \ldots N-1 \longrightarrow (2)$$

Let

$$X_3(K) = X_1(K) \cdot X_2(K) \longrightarrow (3)$$

Let $x_3(m)$ be the seqn whose DFT is $X_3(K)$

Then

$$x_3(m) = \frac{1}{N} \sum_{K=0}^{N-1} X_3(K) \, e^{\frac{j2\pi Km}{N}} \longrightarrow (4)$$

substituting eq (3) in (4) we get

$$x_3(m) = \frac{1}{N} \sum_{K=0}^{N-1} X_1(K) \cdot X_2(K) \, e^{j\frac{2\pi}{N} Km} \longrightarrow (5)$$

substituting (1) & (2) in eq (4)

$$x_3(m) = \frac{1}{N} \sum_{K=0}^{N-1} \left[\sum_{n=0}^{N-1} x_1(n) \, e^{-j\frac{2\pi}{N} Kn} \right] \left[\sum_{l=0}^{N-1} x_2(l) \, e^{\frac{-j2\pi Kl}{N}} \right] e^{\frac{j2\pi Km}{N}}$$

here all the 3 summations have different indices since they are independent. Rearranging the summations & terms in above eqn as follows: 18

$$x_3(m) = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \cdot \left\{ \sum_{k=0}^{N-1} e^{j\frac{2\pi k}{N}(m-n-l)} \right\} \longrightarrow (5)$$

Now let us consider the standard eqn

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & a=1 \\ \dfrac{1-a^N}{1-a} & a \neq 1 \end{cases} \longrightarrow (6)$$

here now $a = e^{j\frac{2\pi}{N}[m-n-l]}$

when $(m-n-l) =$ multiple of N i.e. $N, 2N, 3N$

Then $a = 1$ $\quad \because e^{j\frac{2\pi}{N}N} = e^{j\frac{2\pi}{N}2N}$

$$= e^{j\frac{2\pi}{N}3N} = 1$$

Thus for the 1st condition, we can write. i.e. for $a=1$

$$\sum_{k=0}^{N-1} a^k = \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k[m-n-l]} = N, \text{ when } (m-n-l) \text{ is integer multiple of } N \longrightarrow (7)$$

Now let us consider the 2nd condition. i.e. for $a \neq 1$

The

$$\sum_{k=0}^{N-1} a^k = \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k[m-n-l]} \text{ when } (m-n-l) \text{ is not integer multiple of } N$$

$$= \frac{1-a^N}{1-a}$$

$$= \frac{1 - e^{j\frac{2\pi}{N}[m-n-l]}}{1 - e^{j\frac{2\pi}{N}(m-n-l)}}$$

$$e^{j2\pi(m-n-l)} = 1 \text{ always}$$

(19)

$$\therefore \sum_{k=0}^{N-1} e^{\frac{j2\pi k(m-n-l)}{N}} = 0, \quad \text{when } (m-n-l) \text{ is not multiple of } N.$$

Thus

$$\sum_{k=0}^{N-1} e^{\frac{j2\pi k(m-n-l)}{N}} = \begin{cases} N, & \text{when } (m-n-l) \text{ is multiple of } N \\ 0, & \text{o.w} \end{cases} \rightarrow (8)$$

substituting these in eq (5)

$$x_3(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \cdot N \quad \text{when } (m-n-l) \text{ is integer multiple of } N$$

$$= \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \rightarrow (9)$$

$(m-n-l)$ is a multiple of N

$\therefore$ we can write $(m-n-l) = pN$.

p is an integer & may be +ve or -ve

we can write for our convenience as

$$m-n-l = -pN$$

$$l = m-n+pN \rightarrow (10)$$

sub in eq (9)

$$= \sum_{n=0}^{N-1} x_1(n) \cdot x_2(m-n+pN) \rightarrow (10)$$

since $x_2(m-n+pN)$ is a periodic seq^n with period N.

here $x_2(m)$ is shifted circularly by 'n' samples

$$x_2(m-n+pN) = x_2(m-n, \text{modulo } N)$$
$$\text{or} = x_2((m-n))_N$$

∴ eq (10) becomes

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N, \quad m=0, \ldots N-1 \longrightarrow (11)$$

Let us compare eq (11) with linear conv^n

$$y(n) = \sum_{K=-\infty}^{\infty} x(K)\, h(n-K)$$

eq (11) appears like convolution oper^n but the seq^n is shifted circularly hence it is called circular convolution.

$$\boxed{x_3(m) = x_1(n) \, (N) \, x_2(n) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N}$$

① find the circular convolution using DFT & IDFT for the foll seq^n — use convol^n property

$$x_1(n) = \{2, 3, 1, 1\}$$
$$x_2(n) = \{1, 3, 5, 3\}$$

Sol

$$X_1(K) = \sum_{n=0}^{N-1} x_1(n)\, e^{-j\frac{2\pi}{N}kn}$$
$$= \{7,\ 1-j2,\ -1,\ 1+j2\}$$

$$X_2(K) = \{12,\ -4,\ 0,\ -4\}$$

$y(K) = Y_1(K) \cdot Y_2(K) = \{84, -4+j8, 0, -4-j8\}$

$y(0) = Y_1(0) \cdot Y_2(0)$

(21)

Now take IDFT on $y(K)$

$$y(n) = \frac{1}{N} \sum_{K=0}^{N-1} Y(K) e^{j\frac{2\pi}{N}kn}$$

$$= \{19, 17, 23, 25\}$$

$$y(n) = x_1(n) \, (4) \, x_2(n)$$

(2) given the seq^n $x_1(n) = \{1, 2, 3, 1\}$ & $x_2(n) = \{4, 3, 2, 2\}$. find $y(n)$ such that $y(K) = X_1(K) \cdot X_2(K)$

Sol given $x_1(n) = \{1, 2, 3, 1\}$

$x_2(n) = \{4, 3, 2, 2\}$

$Y(K) = X_1(K) \cdot X_2(K)$

Taking IDFT

$y(n) = x_1(n) \, (N) \, x_2(n)$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 17 \\ 19 \\ 22 \\ 19 \end{bmatrix}$$

Modulation property (or) Multiplication of 2 seqns

U.K

22

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(K)$

& $x_2(n) \xleftrightarrow[N]{DFT} X_2(K)$

Then $y(n) = x_1(n) \cdot x_2(n) \xleftrightarrow[N]{DFT} Y(K) = \frac{1}{N}\left[X_1(K) \, \textcircled{N} \, X_2(K)\right]$

proof: consider 2 seqn $x_1(n)$ & $x_2(n)$

$$x_1(n) \triangleq \frac{1}{N}\sum_{K=0}^{N-1} X_1(K)\, e^{j\frac{2\pi}{N}Kn} \longrightarrow (1) \quad [\text{Kl}(K=m)]$$

$0 \le n \le N-1$

$$x_2(n) \triangleq \frac{1}{N}\sum_{K=0}^{N-1} X_2(K)\, e^{j\frac{2\pi}{N}Kn} \longrightarrow (2) \quad (K = l)$$

$0 \le n \le N-1$

Let $y(n) = x_1(n)\cdot x_2(n) \longrightarrow (3)$

The N-point DFT of $y(n)$ is

$$Y(K) = \sum_{n=0}^{N-1} y(n)\, e^{-j\frac{2\pi}{N}Kn} \longrightarrow (4)$$

sub eq (3) in (4)

$$Y(K) = \sum_{n=0}^{N-1} x_1(n)\cdot x_2(n)\, e^{-j\frac{2\pi}{N}Kn} \longrightarrow (5)$$

sub eq (1) & (2) in eq (5)

$$Y(K) = \frac{1}{N^2}\sum_{n=0}^{N-1}\left[\sum_{K=0}^{N-1} X_1(K)e^{j\frac{2\pi}{N}Kn}\right]\left[\sum_{K=0}^{N-1} X_2(K)e^{j\frac{2\pi}{N}Kn}\right]e^{-j\frac{2\pi}{N}Kn} \longrightarrow (6)$$

Replace K by m & l in eq (1) & (2)

$$= \frac{1}{N^2}\sum_{n=0}^{N-1}\left[\sum_{m=0}^{N-1} X_1(m)e^{j\frac{2\pi}{N}mn}\right]\left[\sum_{l=0}^{N-1} X_2(l)e^{j\frac{2\pi}{N}ln}\right]e^{-j\frac{2\pi}{N}Kn}$$

$$= \frac{1}{N^2}\sum_{m=0}^{N-1} X_1(m)\sum_{l=0}^{N-1} X_2(l)\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}n[K-m-l]} \longrightarrow (7)$$

The last summation in eq (7)

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^n}{1-a}, \quad a \neq 1$$

$N, \quad a = 1$

$$\sum_{n=0}^{N-1} a^n = N, \quad (k-m-l) = PN$$

$= 0, \quad$ o.w

$\therefore$ eq (7) become

$$Y(K) = \frac{1}{N} \sum_{m=0}^{N-1} X_1(m) \sum_{l=0}^{N-1} X_2(l)$$

W.K.T $\quad k-m-l = -PN \quad l = k-m+PN$

$$= \frac{1}{N} \sum X_1(m) \; X_2((K-m))_N$$

$$= \frac{1}{N} [X_1(K) \; (N) \; X_2(K)]$$

(1) Let seqns $x_1(n) = \{1, 1, 2, 1\}$ & $x_2(n) = \{1, 0, 1, 0\}$ find 4-point DFT of $y(n) = x_1(n) \cdot x_2(n)$

Sol find $X_1(K) = \{5, -1, 1, -1\}$

$X_2(K) = \{2, 0, 2, 0\}$

$y(n) = x_1(n) \cdot x_2(n)$

using modulation property $\quad Y(K) = \frac{1}{4} [X_1(K) \; (4) \; X_2(K)]$

$$= \frac{1}{4} \begin{bmatrix} 5 & -1 & 1 & -1 \\ -1 & 5 & -1 & 1 \\ 1 & -1 & 5 & 1 \\ -1 & 1 & -1 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

2. EFFICIENT Computation of FFT

2.1 Introduction:

WKT, DFT is widely used DSP algorithm. DFT also plays an important role in many applications of DSProcessing including linear-filtering, spectrum analysis etc. A major reason for ~~this~~ its importance is the existence of efficient algorithms for computation of DFT

The different computationally efficient Algorithms are discussed here for the evaluating the DFT.

There are 2 approaches for evaluating the DFT in a computationally efficient manner:

(i) divide & conquer approach:

* In this approach, the N-point DFT is reduced to the computation of smaller DFTs from which the larger DFT is computed ~~to here the~~ c. (N- composite no)
* The computational algorithm called Fast-Fourier Transforms (FFT) algorithms for computing the DFT where the size N is power of 2. This is known as radix-2 FFT algorithms.

(ii) A linear filtering approach:

* It is based on the formulation of DFT as a linear filtering opern on the data.
* In this approach the algorithms like ~~Gret~~ ~~Goertzel~~ Goertzel & Chirp-Z transform are used

9.2 Efficient computation of the DFT.

Direct computation of DFT

From the def^n of DFT, we have

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \rightarrow (1)$$

$$0 \le K \le N-1$$

$x(n)$ may be real / complex

$W_N \rightarrow$ Twiddle factor which is complex no.

The above summation includes multiplications & summing of complex no's

~~$X(K) = x(0) W_N^0 + x(1) W_N^K$~~

complex addition

$$X(K) = x(0) \boxed{X} W_N^0 + x(1) \boxed{X} W_N^K + x(2) \boxed{X} W_N^{2K} + \ldots$$

$$\ldots + x(N-1) \boxed{X} W_N^{(N-1)K}$$

complex multiplications

for any values of 'K', we have
$N \rightarrow$ no of complex multiplications
& $(N-1) \rightarrow$ " " " additions $\rightarrow (1)$

$\therefore$ For evaluation of $X(K)$ from 0 to $N-1$ requires

no of complex multiplications $= N \times N = N^2$
no of complex additions $= (N-1) \times N = N(N-1)$
$\rightarrow (2)$

2

Also eq ① can be written as if

$$x(n) = x_R(n) + j\,x_I(n)$$

$$W_N^{kn} = Re(W_N^{kn}) + j\,Im(W_N^{kn})$$

$$X(K) = \sum_{n=0}^{N-1} \{x_R(n) + j\,x_I(n)\}\{Re(W_N^{kn}) + j\,Im[W_N^{kn}]\}$$

$$= \sum_{n=0}^{N-1} x_R(n)\,Re(W_N^{kn}) + j\,x_R(n)\,Im[W_N^{kn}] + j\,x_I(n)\,Re(W_N^{kn}) - x_I(n)\,Im(W_N^{kn})$$

$$= \sum_{n=0}^{N-1} \{x_R(n)\,Re[W_N^{kn}] - x_I(n)\,Im(W_N^{kn})\} + j\,\{x_R(n)\,Im(W_N^{kn}) + x_I(n)\,Re[W_N^{kn}]\}$$

4 real xions

$$= x_R(n) \boxed{\times} Re[W_N^{kn}] - x_I(n) \boxed{\times} Im(W_N^{kn}) + j\,[x_R(n) \boxed{\times} Im(W_N^{kn}) + x_I(n) \boxed{\times} Re(W_N^{kn})]$$

2-real addition

Note:- * subtraction is also counted as a addition in DSP since it requires almost same time as addition.

* $a + jb$ is never executed ∵ it is just a way of representing complex nos

* $\boxed{\text{one complex multiplication is converted into}} \left\{ \begin{array}{l} \to 4 \text{ Real } \times\text{ions} \\ \to 2 \text{ .. } +\text{ions} \end{array} \right. \to (3)$

* for each values of K.

$\boxed{\text{N complex multiplication is converted into}} \left\{ \begin{array}{l} \to 4 \cdot N \text{ Real } \times\text{ion} \\ \to 2 \cdot N \text{ Real } +\text{ion} \end{array} \right. \to (4)$

* K varies from 0 to $N-1$

$\boxed{\text{for complete DFT complex multiplications are converted into}} \left\{ \begin{array}{l} \to 4N^2 = [4N \cdot N] \text{ Real } \times\text{ion} \\ 2N^2 \text{ Real } +\text{ion} \end{array} \right. \to (5)$

* Now let us see how many complex additions are converted into

$(a+jb) \times (c+jd)$
2 complex nos

$$\underbrace{\underbrace{(a+jb)}\ + \underbrace{(c+jd)}}_{} = (a+c) + j(b+d)$$

one complex addition

2 real additions

* $\boxed{\text{one complex addition is converted into}} \to 2 \text{ Real addition} \to (6)$

* for each values of K, there are $(N-1)$ complex additions

$\boxed{\text{hence } (N-1) \text{ complex additions is converted into}} \to 2(N-1) \text{ Real addition} \to (7)$

* K varies from 0 to N-1
then

hence

~~total~~

for complete DFT complex additions are converted into $\Big\}$ $= 2 \times N-1 \times N$

$= 2[N^2 - N]$

$= 2N^2 - 2N$ → (8)

* Total real additions are from eq (5)

Total real additions in computation of DFT $= 2N^2 - 2N + 2N^2$

$= 4N^2 - 2N$

$= N[4N-2]$ → (9)

Thus for direct computation of N-point DFT requires the foll arithmetic operⁿ:

* N complex multiplications for each value of K
* N^2 complex multiplications for all values of K
* (N-1) complex additions for each values of K
* (N-1) complex additions for all values of K
* N(N-1) complex additions

(or)

* 4N real multiplications for each value of K
* $4N^2$ real multiplications for all values of K
* (4N-2) real additions for each values of K

$(2N-2+2N = 4N-2)$

* N(4N-2) real additions for all values of K.

* ∴ The no of arithematic operns in direct computation of DFT is large & thus it is time consuming.
* The total no of operations ↑ses very rapidly as N ↑ses.
* Hence it is of practical interest to develop more efficient fast algorithms for computing the DFT.

Periodicity property of W_N

① $W_N^{K+N} = W_N^K$

Proof: $W_N = e^{-j2\pi/N}$

$W_N^{K+N} = e^{-j\frac{2\pi}{N}[K+N]}$

$= e^{-j\frac{2\pi}{N}k} \cdot \cancel{e^{-j\frac{2\pi}{N}N}} = 1$

$= e^{-j\frac{2\pi}{N}k}$

$= W_N^k$

② $W_N^2 = W_{N/2}$

$W_N = e^{-j\frac{2\pi}{N}}$

replace N by N/2

$W_{\frac{N}{2}} = e^{-j\frac{2\pi}{N/2}}$

$= e^{-j\frac{2\pi}{N}\cdot 2}$

$= W_N^2$

③ Symmetry

$W_N^{K+\frac{N}{2}} = -W_N^K$

Proof:- $W_N = e^{-j\frac{2\pi}{N}}$

$W_N^{K+\frac{N}{2}} = e^{-j\frac{2\pi}{N}[K+N/2]}$

$= e^{-j\frac{2\pi}{N}k} \cdot e^{-j\frac{2\pi}{N}\cdot\frac{N}{2}}$

$= e^{-j\frac{2\pi}{N}k} \cdot e^{-j\pi}$

$e^{-j\pi} = -1$ always

$= -e^{-j\frac{2\pi}{N}k}$

$= -W_N^k$ //

2.3 Radix-2. FFT Algorithm :

* By employing divide & conquer approach, a computationally efficient algorithm to evaluate DFT can be developed.
* This approach depends on the decomposition of N-point DFT into successively smaller size DFTs.

* Let $N = r_1 ; r_2 \cdot r_3 \cdots r_v$

  where $r_1 = r_2 = r_3 = \cdots r_v = r$

  then

  $$N = r^v$$

  here the no r is called radix of FFT algorithm
* here the most widely used is radix-2 FFT algorithms when $r = 2$, ~~as~~ are explained.
* There are 2 types of radix-2 FFT algorithms they are

  (a) Radix-2 : Decimation in Time FFT algorithm [DIT-FFT] algorithm

  (b) Radix-2 : Decimation in Frequency FFT algo. [DIF-FFT] algorithm

9.2 * ## Decimation in Time - FFT Algorithm [DIT-FFT]:-

+ Let us assume $x(n)$ is a length - N sequence & N is assumed as a power of $_2$ V (2^V)

Let V = 2 . = N^2. $N = 2^P$

+ In this algorithm, the N-point DFT is decimated (broken) into two $\frac{N}{2}$-point ~~point~~ DFTs. each $\frac{N}{2}$-point DFT is decimated into two $\frac{N}{4}$-point DFTs & this decimation is continued until 2-point DFTs are obtained

N-point DFT
decimated

$\frac{N}{2}$-point DFTs $\frac{N}{2}$-point DFTs

$\frac{N}{4}$ $\frac{N}{4}$ $\frac{N}{4}$ $\frac{N}{4}$

2-point DFTs are obtained

This approach is called as <u>divide & conquer approach</u>

* ~~consider that the given length N seqⁿ~~

~~$x(n) = \{ x(0), x(1), x(2) \ldots x(\ldots x(N-1) \}$~~

(5)

consider that the given length - N seq^n is

$x(n) = \{x(0), x(1), x(2), \ldots x(\frac{N}{2}-1), \ldots x(N-1)\}$

I-stage:

Decimate this seq^n $x(n)$ into two seq^n of length $N/2$. one considered as even-indexed values of $x(n)$ & other as odd-indexed values of $x(n)$.

Even-indexed seq^n : $\{x(0), x(2), x(4) \ldots, x(N-2)\}$

odd-indexed seq^n : $\{x(1), x(3) \ldots x(N-1)\}$

WKT by def^n of DFT,

$$X(K) \triangleq \sum_{n=0}^{N-1} x(n)\, W_N^{Kn} \longrightarrow (1)$$

$$0 \le k \le N-1$$

decimating $x(n)$ in eq (1) into even & odd indexed seq^{n}s, we get

$$X(K) = \sum_{\substack{n=0 \\ n=\text{even}}}^{N-2} x(n)\, W_N^{Kn} + \sum_{\substack{n=1 \\ n=\text{odd}}}^{N-1} x(n)\, W_N^{Kn} \longrightarrow (2)$$

substituting $n = 2r$ in first summation & $n = 2r+1$ in second summation, we get

$$X(K) = \sum_{r=0}^{\frac{N}{2}-1} x(2r).\, W_N^{2Kr} + \sum_{r=0}^{\frac{N}{2}-1} x(2r+1)\, W_N^{K(2r+1)} \longrightarrow (3).$$

Note: $W_N^{K2r} = W_{N/2}^{Kr}$

$$= \sum_{r=0}^{N/2} g(r)\, W_N^{2Kr} + \sum_{r=0}^{N/2-1} h(r)\, W_N^{2Kr} \cdot W_N^{K}$$

$$= \underbrace{\sum_{r=0}^{\frac{N}{2}-1} g(r)\, W_{\frac{N}{2}}^{kr}}_{\frac{N}{2}\text{-point DFT of even indexed seqn}} + W_N^{k} \underbrace{\sum_{r=0}^{\frac{N}{2}-1} h(r)\, W_{N/2}^{kr}}_{\frac{N}{2}\text{-point DFT of odd-indexed seqn}} \rightarrow (4)$$

if $G(K)$ & $H(K) \rightarrow \frac{N}{2}$-point DFTs of even & odd indexed seqn

$$0 \le K \le \frac{N}{2} - 1$$

& they are also periodic with a period $N/2$

we have

$$\left.\begin{aligned} G(K) &= G\left(K - \frac{N}{2}\right) \\ \& \ H(K) &= H(K - N/2) \end{aligned}\right\} \rightarrow (5)$$

eq (4) can be written as

$$X(K) = G(K) + W_N^{K} H(K) \rightarrow (6)$$

$$0 \le K \le \frac{N}{2} - 1$$

using eq (5) $\therefore$ for $\frac{N}{2} \le K \le N-1$

eq (6) can be written as

$$X(K) = G\left(K - \frac{N}{2}\right) + W_N^{K} H\left(K - \frac{N}{2}\right) \rightarrow (7)$$

$$; \ \frac{N}{2} \le K \le N-1$$

$\therefore$

$$X(K) = \begin{cases} G(K) + W_N^{K} H(K), & 0 \le K \le \frac{N}{2} - 1 \\ G\left(K - \frac{N}{2}\right) + W_N^{K} H\left(K - \frac{N}{2}\right), & \frac{N}{2} \le K \le N-1 \end{cases} \rightarrow (8)$$

(6)

consider $N = 2^{\nu}$ ∴ $N = 8$. Then $0 \le K \le 7$

from eq (8) we get

$$X(K) = \begin{cases} G(K) + W_8^K H(K), & K = 0, 1 \ldots \frac{N}{2} - 1 \\ G(K+\frac{8}{2}) + W_8^K H(K+\frac{8}{2}), & K = \frac{N}{2}, \ldots N-1 \end{cases} \rightarrow (9)$$

$\frac{N}{2} - 1$
$= \frac{8}{2} - 1$
$= 4 - 1$
$= 3$

$$X(K) = \begin{cases} G(K) + W_8^K H(K), & K = 0, 1, 2, 3 \\ G(K-4) + W_8^K H(K-4), & K = 4, 5, 6, 7 \end{cases} \rightarrow (9)$$

$X(0) = G(0) + W_8^0 H(0)$

$X(1) = G(1) + W_8^1 H(1)$

$X(2) = G(2) + W_8^2 H(2)$

$X(3) = G(3) + W_8^3 H(3)$

$X(4) = G(0) + W_8^4 H(0)$

$X(5) = G(1) + W_8^5 H(1)$

$X(6) = G(2) + W_8^6 H(2)$

$X(7) = G(3) + W_8^7 H(3)$.

using these sets of eq^n we obtain the flow graph after 1st stage decimation of 8-point DFT

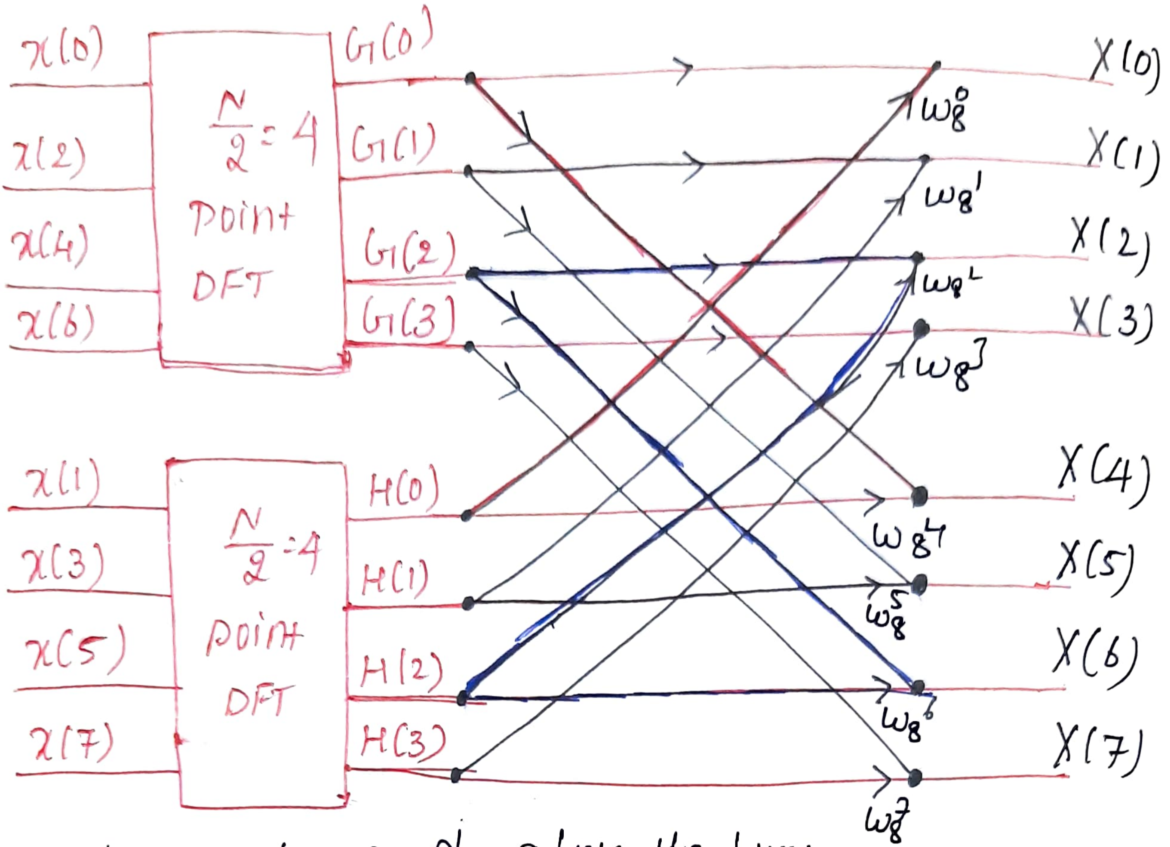


fig ① Flow-graph after the first stage decomposition in DIT-FFT algorithm for N=8.

* In general, the no of complex multiplications required to evaluate the N-point DFT with the first stage decomposition is given by:

$$\alpha = \left(\frac{N}{2}\right)^2 + \left(\frac{N}{2}\right)^2 + N$$

no of complex xions required for direct computation of $\frac{N}{2}$ point DFT $G(K)$

no of complex multiplication required for direct computation of $\frac{N}{2}$ point DFT $H(K)$

no of complex xions required to multiply W_N^K.

$$= 2 \cdot \frac{N^2}{4} + N = \frac{N^2}{2} + N$$

* Thus the no of complex xions is reduced from N^2 to $\frac{N^2}{2} + N$

$8^2 = 64$
$\frac{64}{2} + 8 = 32 + 8$

II nd. stage :- each $\frac{N}{2}$ point seq^n are further decimated into seqn's of length $\frac{N}{4}$.

we have

$$G(K) = \sum_{n=0}^{N/2 - 1} g(r)\, W_{\frac{N}{2}}^{Kr} \longrightarrow (10)$$

decimated $g(r)$ of eq (10) into even & odd-indexed seq^n.

$$G(K) = \sum_{r=0}^{\frac{N}{2}-2} g(r)\, W_{\frac{N}{2}}^{Kr} + \sum_{r=1}^{\frac{N}{2}-1} g(r)\, W_{N/2}^{Kr} \longrightarrow (11)$$

substituting $r = 2l$ in 1st summation & $r = 2l+1$ in 2nd summation we get

$$G(K) = \sum_{l=0}^{\frac{N}{4}-1} g(2l)\, W_{\frac{N}{2}}^{2Kl} + \sum_{l=0}^{\frac{N}{4}-1} g(2l+1)\, \underbrace{W_{\frac{N}{2}}^{K(2l+1)}}_{W_{\frac{N}{2}}^{2Kl} \cdot W_{\frac{N}{2}}^{K}}$$

$$= \sum_{l=0}^{\frac{N}{4}-1} g(2l)\, W_{N/2}^{2Kl} + \sum_{l=0}^{\frac{N}{4}-1} g(2l+1)\, W_{\frac{N}{2}}^{2Kl} \cdot W_{N/2}^{K}$$

$$= \underbrace{\sum_{l=0}^{\frac{N}{4}-1} g(2l)\, W_{\frac{N}{4}}^{Kl}}_{A(K)} + W_{\frac{N}{2}}^{K} \underbrace{\sum_{l=0}^{\frac{N}{4}-1} g(2l+1)\, W_{\frac{N}{4}}^{Kl}}_{B(K)}$$

$$G(K) = A(K) + W_{\frac{N}{2}}^{K} B(K) \longrightarrow (12)$$

$$0 \le K \le \frac{N}{4} - 1$$

$A(K) \rightarrow \frac{N}{4}$ - point DFT of even indexed seq^n of $g(r)$

$B(K) \rightarrow$ " " " " odd " " " $g(r)$

1111'4

$$H(K) = C(K) + W_{N/2}^{K} D(K) \rightarrow (13).$$

since $A(K)$, $B(K)$, $C(K)$, & $D(K)$ are periodic with a period 4 we can write

$$G(K) = \begin{cases} A(K) + W_{\frac{N}{2}}^{K} B(K), & K = 0, 1, \ldots \frac{N}{4}-1, \\ A(K-\frac{N}{4}) + W_{\frac{N}{2}}^{K} B(K-\frac{N}{4}), & K = \frac{N}{4}, \frac{N}{4}+1, \ldots \frac{N}{2}-1 \end{cases} \rightarrow (14)$$

&

$$H(K) = \begin{cases} C(K) + W_{\frac{N}{2}}^{K} D(K) & \ldots K = 0, 1, \ldots \frac{N}{4}-1 \\ C(K-\frac{N}{4}) + W_{\frac{N}{2}}^{K} B(K-\frac{N}{4}), & K = \frac{N}{4} \ldots \frac{N}{2}-1 \end{cases} \rightarrow (15)$$

for $N=8$.

$$G(K) = \begin{cases} A(K) + W_4^{K} B(K), & K = 0, 1 \\ A(K-2) + W_4^{K} B(K-2), & K = 2, 3 \end{cases}$$

$$H(K) = \begin{cases} C(K) + W_4^{K} B(K), & K = 0, 1 \\ C(K-2) + W_4^{K} D(K-2), & K = 2, 3 \end{cases}$$

$$G(0) = A(0) + W_4^{0} B(0)$$
$$G(1) = A(1) + W_4^{1} B(1)$$
$$G(2) = A(0) + W_4^{2} B(0)$$
$$G(3) = A(1) + W_4^{3} B(1)$$
$$H(0) = C(0) + W_4^{0} D(0)$$
$$H(1) = C(1) + W_4^{1} D(1)$$
$$H(2) = C(0) + W_4^{2} D(0)$$
$$H(3) = C(1) + W_4^{3} D(1)$$

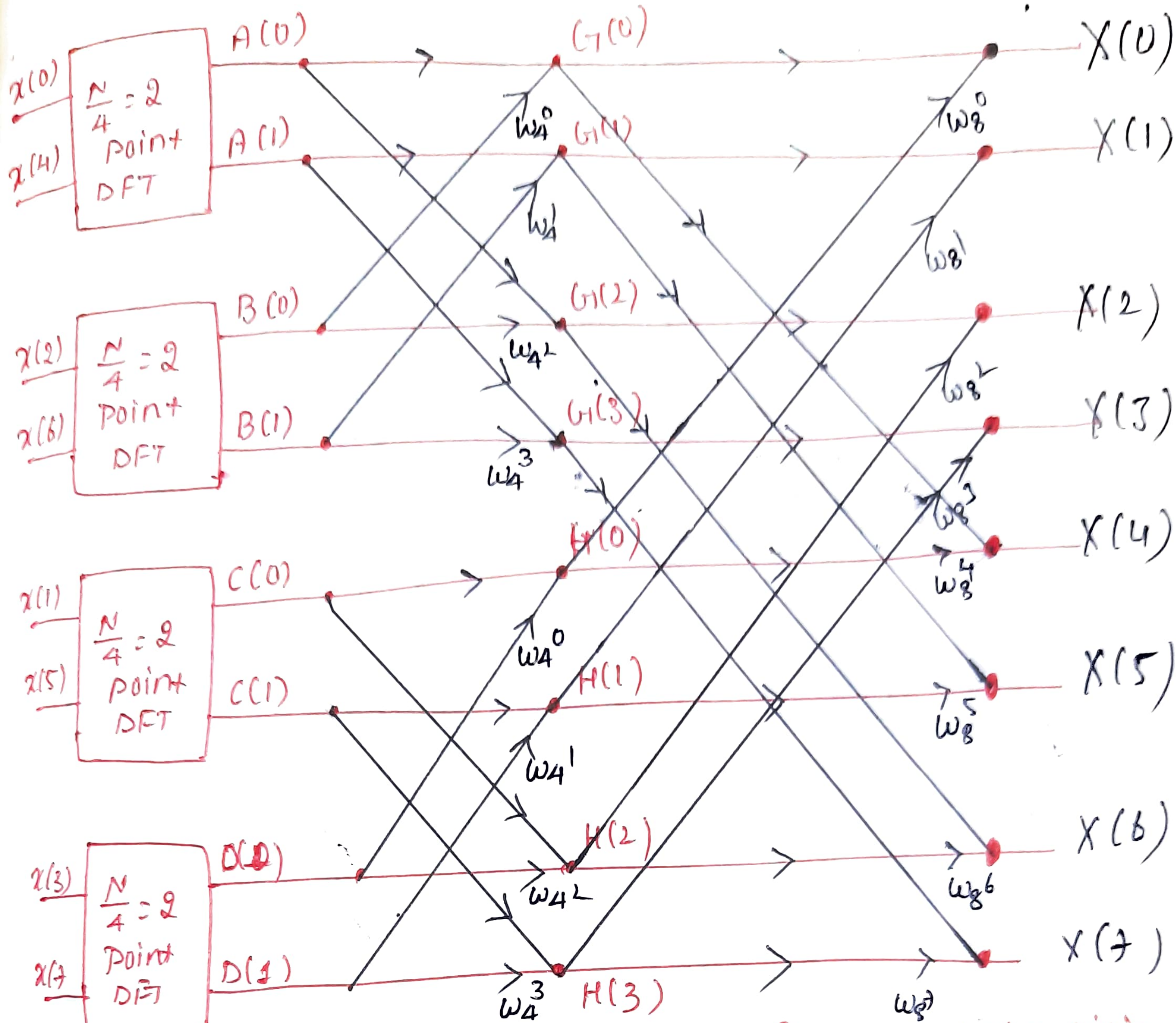


fig ② flow-graph after the 2nd stage decomposition in DIT-FFT Algorthim for N=8

* In general, the no of complex multiplications required after 2nd stage decimation is given below.

no of x'ions required to multiply W_N^k

$$\beta = \underbrace{\left(\frac{N}{4}\right)^2 + \left(\frac{N}{4}\right)^2 + \left(\frac{N}{4}\right)^2 + \left(\frac{N}{4}\right)^2}_{\text{no of complex x'ions required for direct computation of four } \frac{N}{4} \text{ point DFT}} + \underbrace{\left(\frac{N}{2}\right) + \left(\frac{N}{2}\right)}_{\text{no of complex x'ions required for multiply the factor } W_{N/2}^k} + N$$

$$= 4\left(\frac{N}{4}\right)^2 + 2\left(\frac{N}{4}\right) + N$$

$$\boxed{\beta = \frac{N^2}{4} + 2N}$$

* Continuing this process of decimation, we can represent each $\frac{N}{4}$-point DFT as a combination of two $\frac{N}{8}$-point DFT & so..on.

* N is a power of 2, i.e., $N = 2^V$ this process is continued until there are $V = \log_2^N$ stages.

* In the above example for $N=8$, the 8-point DFT computation has been reduced to a computation of 2-point DFTs after the second stage decimation.

* Let us consider the 2-point DFT of $x(0)$ & $x(4)$ we have

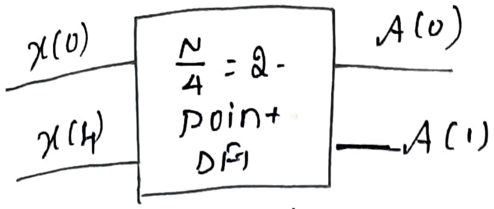
$$A(K) = \sum_{n=0}^{\frac{N}{4}-1} x(n)\, W_{N/4}^{kn} \quad ; \quad 0 \leq K \leq 1$$

$$= \sum_{n=0}^{1} x(n)\, W_{N/4}^{kn}$$

$$\cancel{A(0) = x(0)\, W_{N/4}^{0}}$$

$$A(K) = x(0)\, W_{N/4}^{0} + x(1)\, W_{N/4}^{n}$$

$$\cancel{A(0) = x(0) + x(4)}\, W_{N/}$$



can be written as in fig 3

$$A(K) = \sum_{n=0}^{1} x(n)\, W_{\frac{N}{4}}^{kn}$$

$$= x(0)\, W_{\frac{N}{4}}^{0} + x(1)\, W_{\frac{N}{4}}^{k}$$

$$A(K) = x(0) + x(4)\, W_2^{K}$$

$$A(0) = x(0) + W_2^0\, x(4) \longrightarrow (16)$$

$$A(1) = x(0) + W_2^1\, x(4) \longrightarrow (17)$$

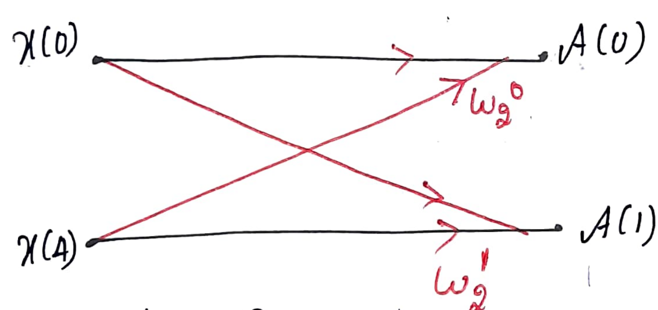
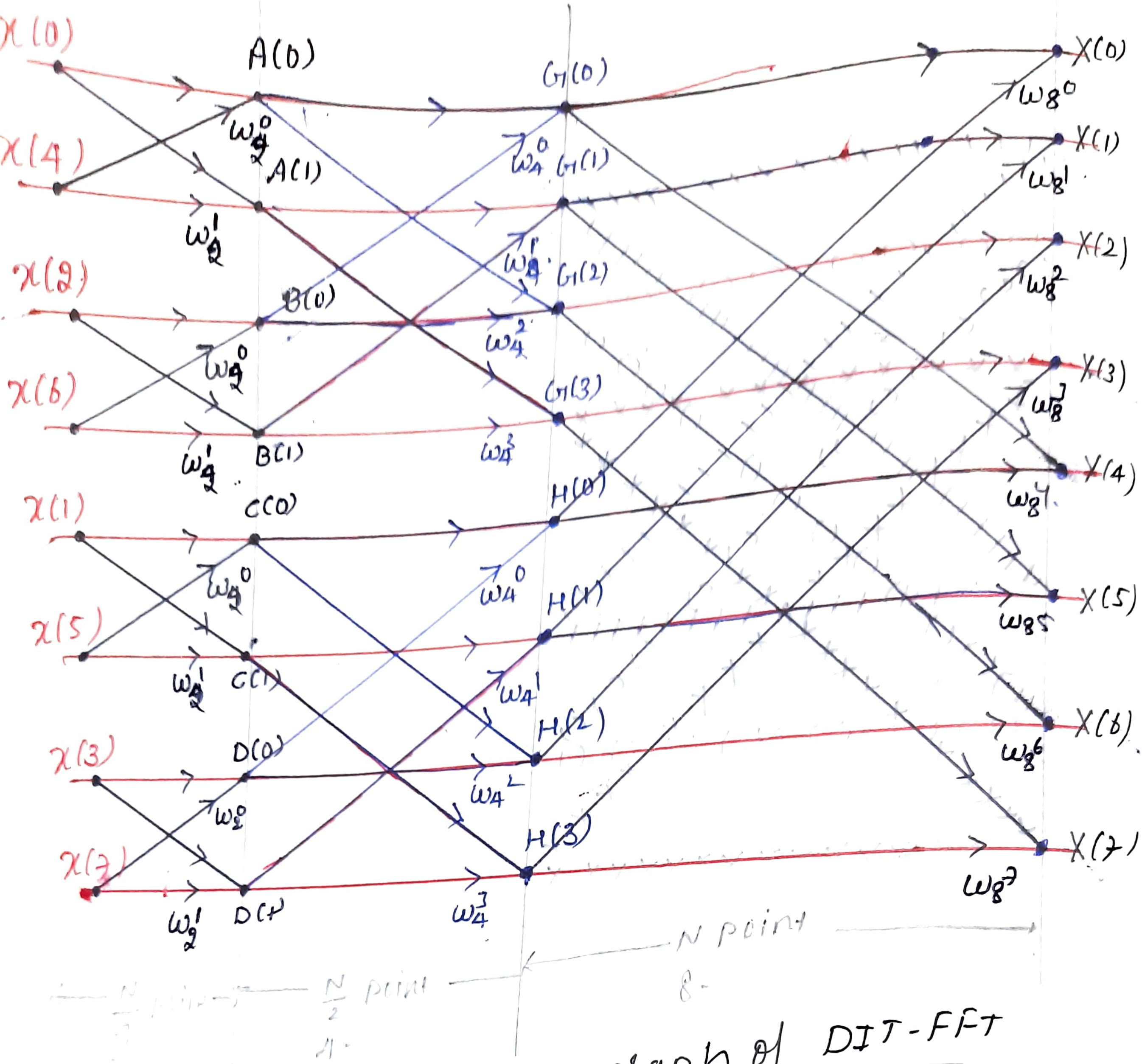


fig 3.

Inserting the flow-graph shown in fig 3 in fig 2 for computation of
The complete flow-graph
8-point DFT is as shown in fig 4



fig④ The complete flow graph of DIT-FFT Algorithm for N = 8

from the flow graph shown above,

* for each stage there are 8 complex multiplications & 8 complex additions.
* In general for computation of N-point DFT using DIT-FFT Alg, there are N complex multiplication & N complex additions for each stance
* since there are $V = \log_2 N$ stages ∴ A total of ∴ $N \log_2 N$ complex xlions & +ions are required to compute the N-point DFT using DIT-FFT Alg

* In-place computation :-

from fig (4) we have the foll points :

(a) The input data appear in 'bit reversed' order as illustrated below

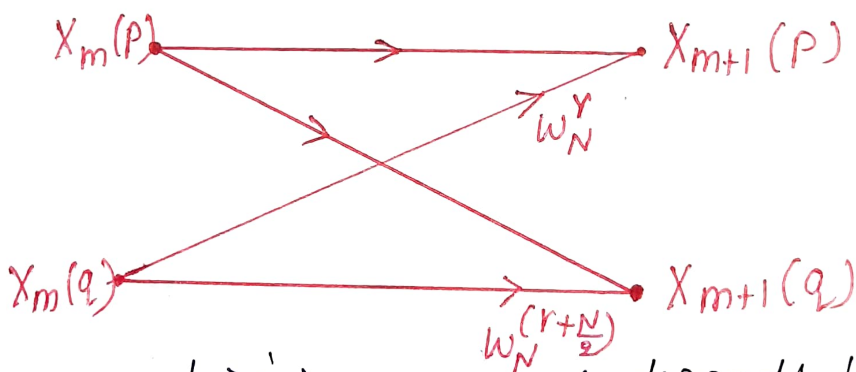
	bit reversal
000	000 – 0
001	100 – 4
010	010 → 2
011	110 → 6
100	001 → 1
101	101 → 5
110	011 → 3
111	111 → 7

(b) Each basic computational block in the diagram is called a 'butterfly' because of its diagram

(c) The N-point DFT X(K) are appears in the normal order ((or) freq domain o/p)

* ~~if we~~ let $m \rightarrow$ represent the stage
$p \& q \rightarrow$ position no in the stages

each butterfly in fig (4) can be represented as shown in fig (5) below



* The value of 'r' is variable & depends upon the position of the butterfly.

* The o/ps $X_{m+1}(P)$ & $X_{m+1}(q)$ of the buttefly at stage (m+1) are calculated in terms of $X_m(P)$ & $X_m(q)$ which are the o/p values from the mth stage & no other i/ps

$$X_{m+1}(P) = X_m(P) + W_N^r \cdot X_m(q) \longrightarrow 18$$

$$X_{m+1}(q) = X_m(P) + -W_N^{(r+\frac{N}{2})} X_m(q) \longrightarrow 19$$

* This kind of computation is known as in-place computation.

* Further Reduction :- (using Cooley-Tukey algorithm)

from eq (18) & (19) we have

$$X_{m+1}(p) = X_m(p) + W_N^r X_m(q)$$

$$X_{m+1}(q) = X_m(p) + W_N^{(r+\frac{N}{2})} X_m(q)$$

Also we have

$$W_N^{(r+\frac{N}{2})} = W_N^r \cdot W_N^{N/2}$$

$$W_N^{N/2} = e^{-j\frac{2\pi}{N}\cdot\frac{N}{2}} = e^{-j2\pi} = -1 \text{ always}$$

$\therefore$

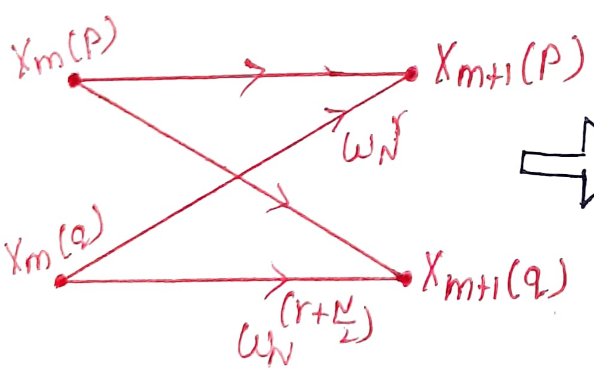
$$W_N^{(r+N/2)} = -W_N^r \rightarrow (20)$$

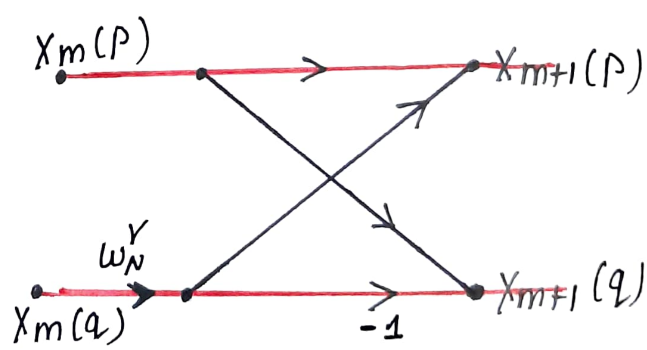
$\therefore$ eq (18) & (19) can be written as

$$X_{m+1}(p) = X_m(p) + W_N^r X_m(q) \rightarrow (21)$$

$$X_{m+1}(q) = X_m(p) - W_N^r X_m(q) \rightarrow (22)$$

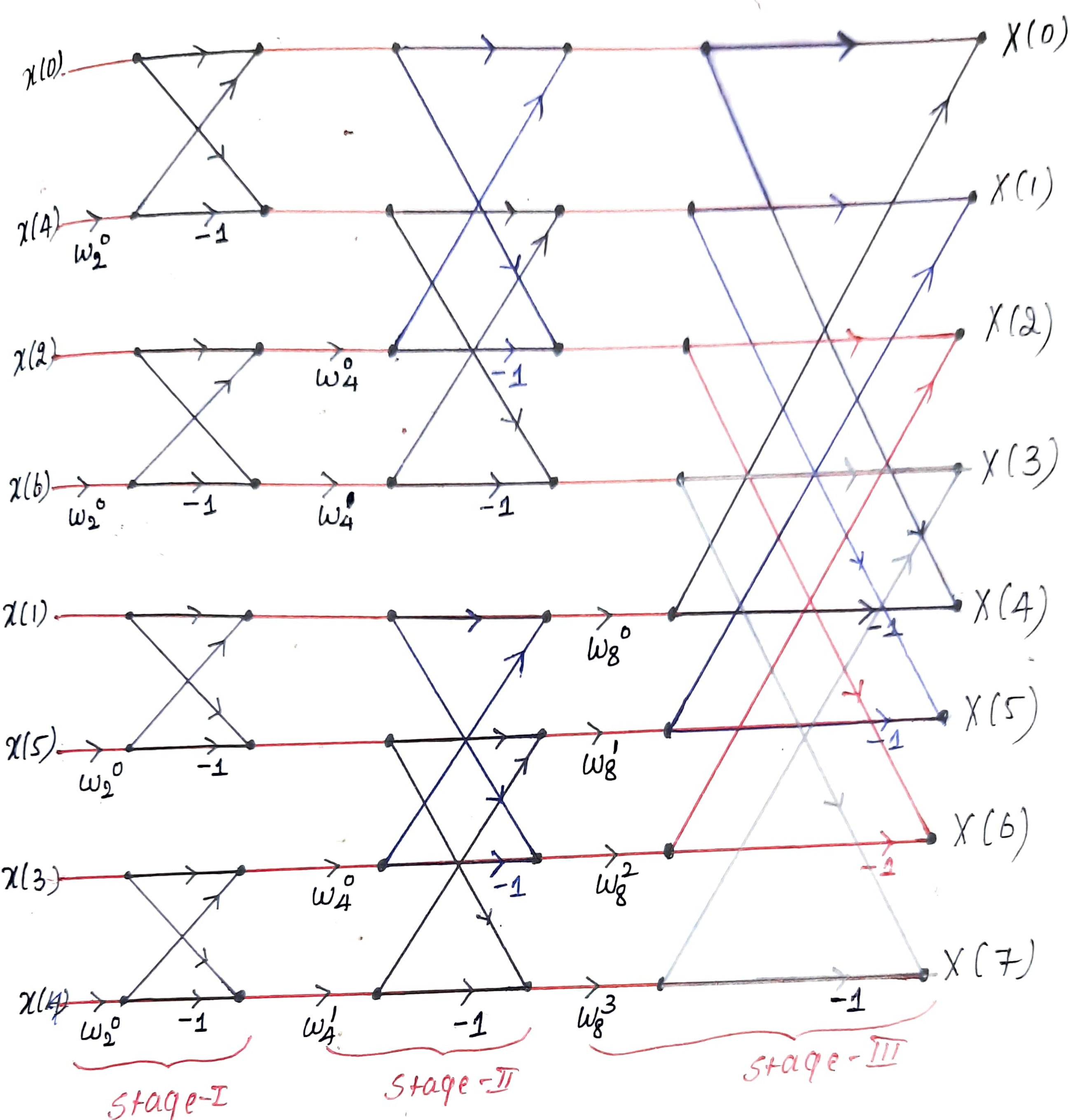
using eq (21) & (22) the buttefly shown in fig (5) can be modified as shown in fig (6).





* After further reduction the total no of complex multiplications are reduced to $\frac{N}{2} \times \log_2 N$

* The reduced complete 8-point DIT-FFT Alg is shown in fig (7) below



8-point DIT-FFT flow graph

$$\boxed{\text{Note:}\quad W_2^0 = W_4^0 = W_8^0 \qquad W_4^1 = W_8^2}$$

$W_2^0 = 1$

$W_4^0 = 1$

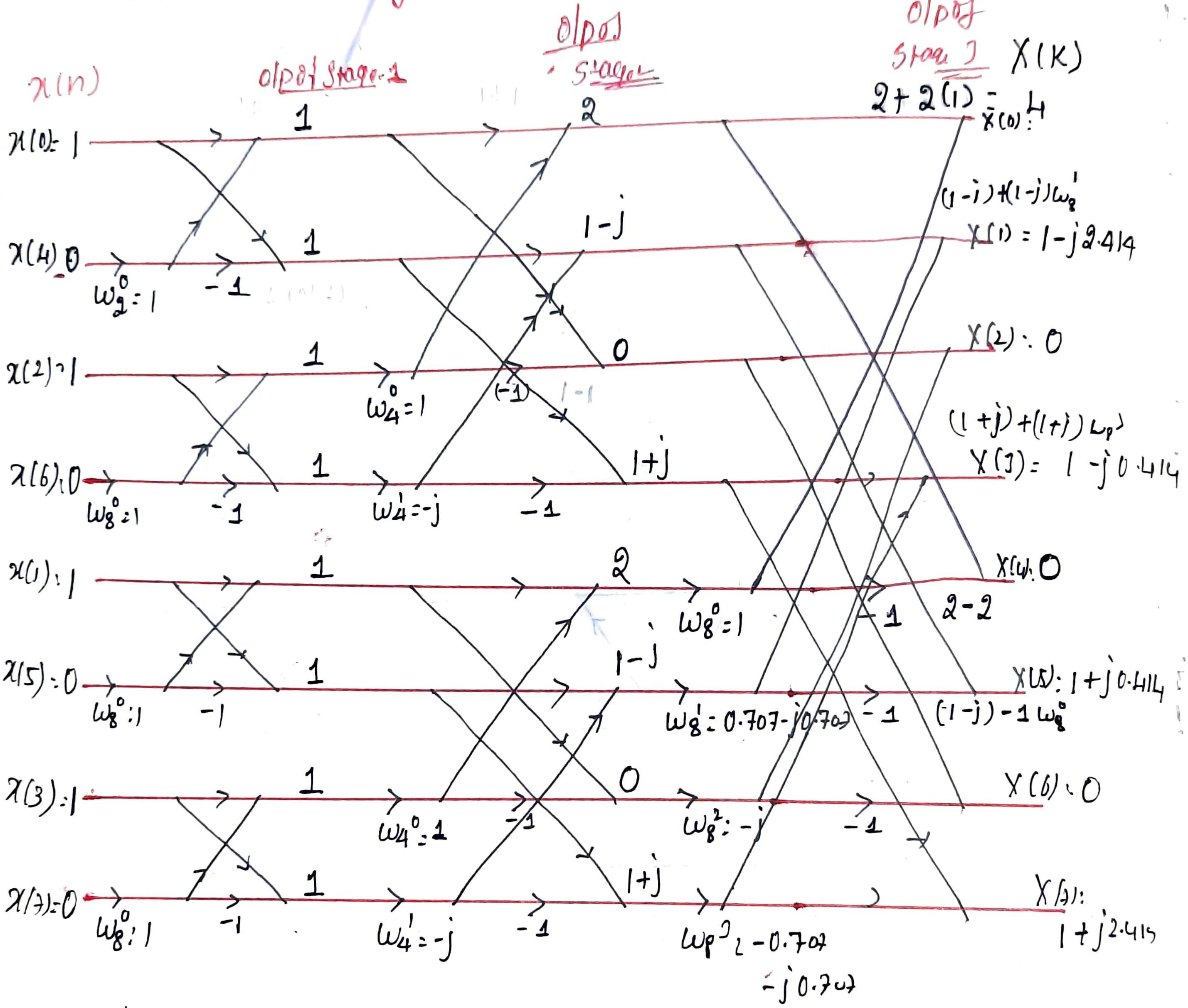
$W_8^0 = 1$

$W_4^1 = -j$

$W_8^1 = 0.707 - j0.707$

$W_8^2 = -j$; $W_8^3 = -0.707 - j0.707$

① Compute 8-point DFT of the seq^n $x(n)$ using $x(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$ using DIT-FFT Algorithm.



$$
\begin{array}{r} 0.707 \\ 0.707 \\ +1 \\ \hline 2.414 \end{array}
$$

$(1-j) + (0.707 - j0.707)(1-j)$

$1 - j + 0.707 - j0.707 - 0.707j - 0.707$

$1 - j2.414$

$X(K) = \{4, 1 - j2.414, 0, 1 - j0.414, 0, 1 + j0.414, 0, 1 + j2.414\}$

2) Find the 8-point DFT of the seqn $x(n) = 2n$, $0 \le n \le 7$, use DIT-FFT Alg.

Soln.

$x(n) = \{0, 2, 4, 8, 16, 32, 64, 128\}$

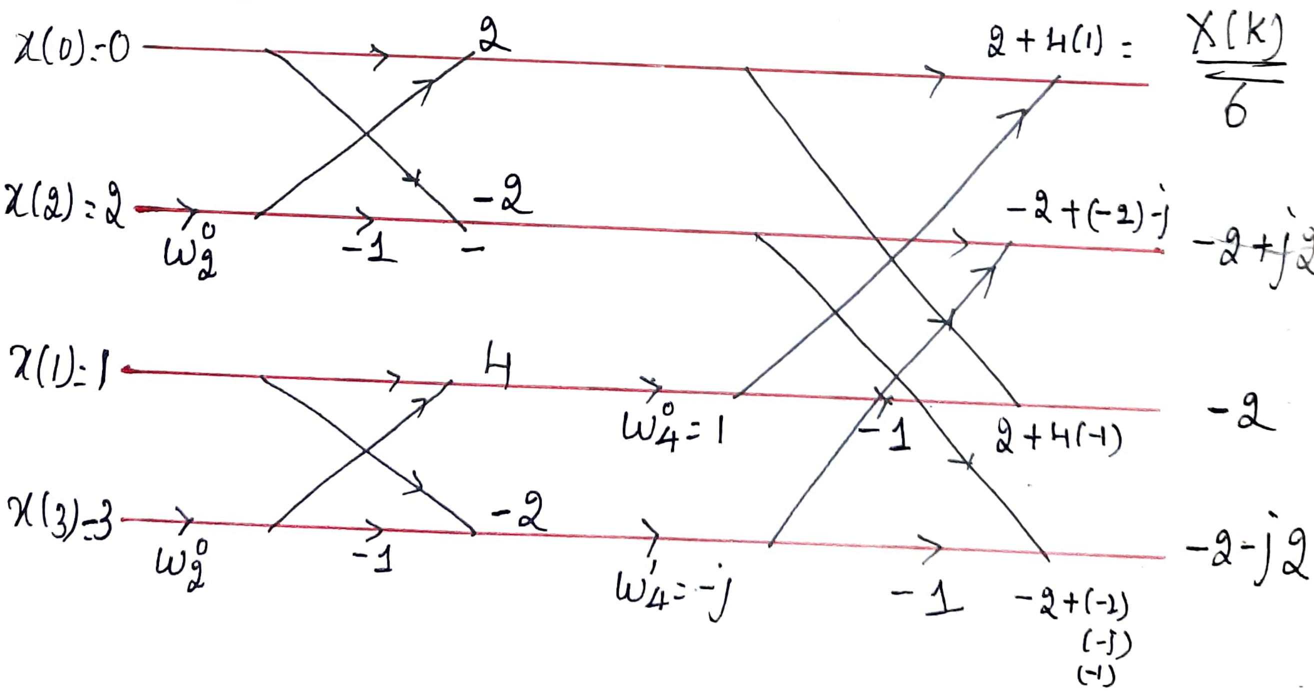
O/p of I-Stage $= \{17, -15, 68, -60, 34, -30, 136, -120\}$

O/p of II-Stage $: \{85, -15+j60, -51, -15-j60, 170, -30+j120, -102, -30-j120\}$

O/p of III Stage

$X(K): \{255, 48.63+j166.05, -51+j102, -78.63+j46.05, -85, -78.63-j46.05, -51-j102, 48.63-j166.05\}$

3) Given $x(n) = \{0, 1, 2, 3\}$ find its DFT using DIT-FFT Alg. $N=4$



$\therefore X(K) = \{6, -2+j2, -2, -2-j2\}$

[Ex.]

Decimation in frequency
DIF-FFT Algorithm :-

~~The IA~~ The DIT algorithm is based on the decomposition of the DFT computation by forming smaller & smaller subsequences of seq^n $x(n)$.
where as in DIF-algorithm, the o/p seq^n $X(K)$ is divided into smaller & smaller subseq^ns.

I-stage :- The decimation in freq-FFT Algorithm is obtained by breaking $X(K)$ as follows:

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn} \quad ; \quad 0 \le K \le N-1 \longrightarrow ①$$

spliting the i/p seq^n in to a length of $\frac{N}{2}$ samples

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{Kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n)\, W_N^{Kn}.$$

substitute ~~let~~ $r = n - \frac{N}{2}$ in second summation

$\therefore n = r + \frac{N}{2}$

$r = n - \frac{N}{2}$
$n = \frac{N}{2},\ r = 0$
$n = N-1\ \ r =$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{Kn} + \sum_{r=0}^{\frac{N}{2}-1} x\left(r+\frac{N}{2}\right) \cdot W_N^{K\left(r+\frac{N}{2}\right)}$$

since r is a dummy variable, it can be replaced by n we get

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{Kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) \cdot W_N^{K\left(n+\frac{N}{2}\right)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{Kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) \cdot W_N^{Kn} \cdot W_N^{K \cdot N/2} \longrightarrow ②$$

WKT $W_N^{KN/2} = e^{-\frac{j2\pi}{N}.K.\frac{N}{2}} = e^{-j2\pi} = (-1)^K$

$\therefore$ eq (2) reduces to

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{Kn} + \sum_{n=0}^{\frac{N}{2}-1} x(n+\tfrac{N}{2})\, W_N^{Kn}\, (-1)^K$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^K x(n+\tfrac{N}{2})\right] W_N^{Kn} \longrightarrow (3)$$

The decimation in freq is now obtained by getting even & odd terms of $X(K)$.

(i) even term: substitute $\cancel{K=}$ $2r$ in eq (3)

$\cancel{X(K) = \sum}$

$$X(2r) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^{2r} x(n+\tfrac{N}{2})\right] W_N^{2rn}$$

$$\left[(-1)^{2r} = 1 \ \forall r \right.$$

$$\left. \& \quad W_N^{2rn} = W_{N/2}^{rn} \right]$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + x(n+\tfrac{N}{2})\right] W_{N/2}^{rn}$$

$$\boxed{X(2r) = \sum_{n=0}^{N/2-1} g(n)\, W_{\frac{N}{2}}^{rn} \qquad 0 \leq r \leq \tfrac{N}{2}-1} \longrightarrow (4)$$

$|||^{ly}$ $K = 2r+1$ in eq ③

$$X(2r+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^{2r+1} x\left(n+\frac{N}{2}\right)\right] W_N^{(2r+1)n}$$

$$(-1)^{2r+1} = -1 \quad \forall r$$

$$W_N^{(2r+1)n} = W_N^{2rn} \cdot W_N^{n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_{\frac{N}{2}}^{rn} \cdot W_N^{n}$$

or

$$\boxed{X(2r+1) = \sum_{n=0}^{N/2-1} h(n)\, W_{\frac{N}{2}}^{rn} \; ; \quad 0 \le r \le \frac{N}{2}-1} \rightarrow ⑤$$

$$\boxed{\begin{aligned} g(n) &= x(n) + x\left(n+\frac{N}{2}\right) \\ h(n) &= \left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_N^{n} \end{aligned}} \rightarrow ⑥$$

for $N = 8$ eq ⑥ becomes

$g(n) = x(n) + x(n+4)$

$g(0) = x(0) + x(4)$

$g(1) = x(1) + x(5)$

$g(2) = x(2) + x(6)$

$g(3) = x(3) + x(7)$

$(n = 0 \text{ to } 3)$

$h(n) = [x(n) - x(n+4)]\, W_8^{n}$

$h(0) = [x(0) - x(4)]\, W_8^{0}$

$h(1) = [x(1) - x(5)]\, W_8^{1}$

$h(2) = [x(2) - x(6)]\, W_8^{2}$

$h(3) = [x(3) - x(7)]\, W_8^{3}$

(14)

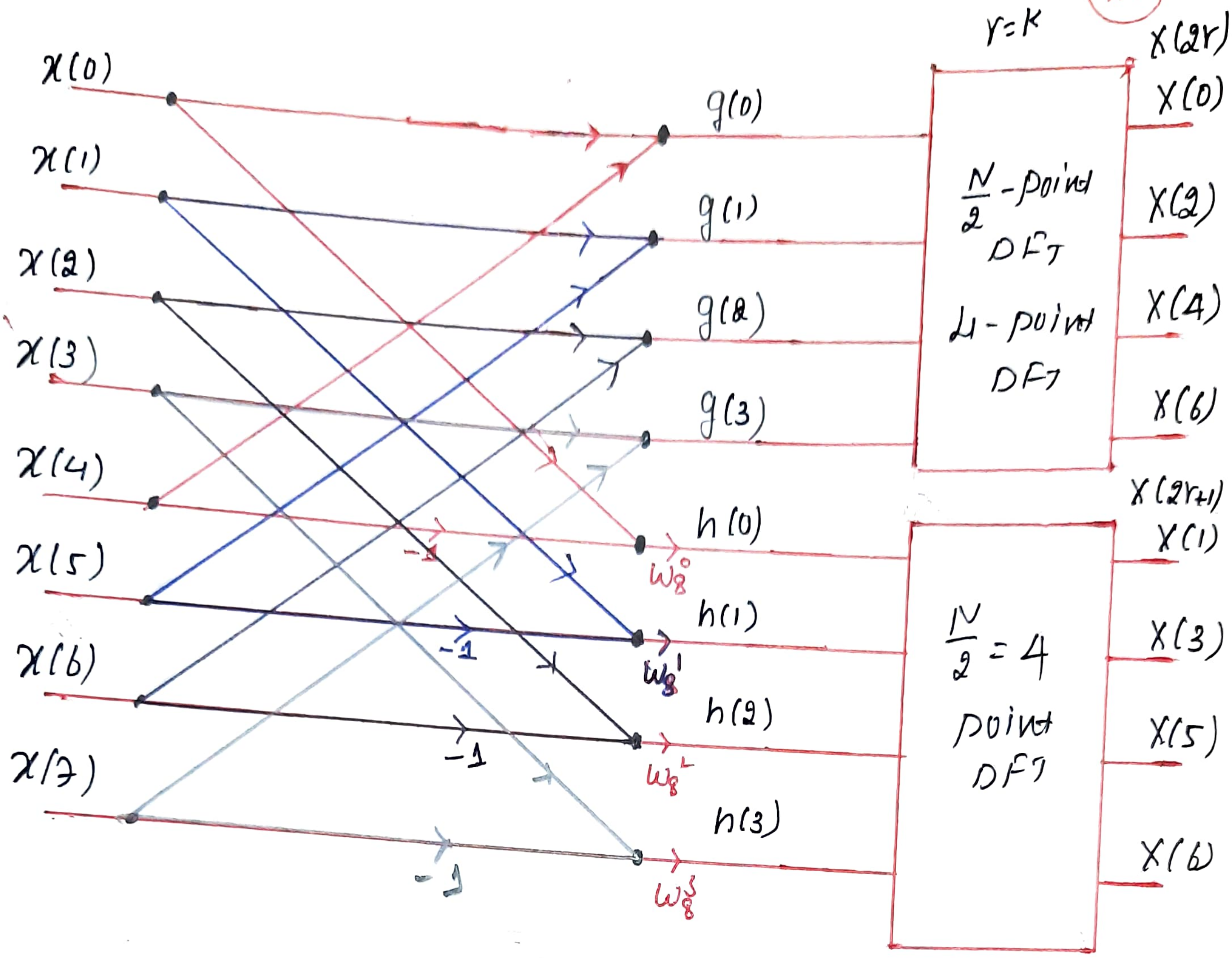


fig ① : signal Flow graph after 1st stage decimation of DIF-FFT Alg for N=8.

* no of complex xlions = $2 \cdot \left(\frac{N}{2}\right)^2$

II - stage decimation

consider eq ④

$$X(2r) = \sum_{n=0}^{\frac{N}{2}-1} g(n)\, W_{\frac{N}{2}}^{rn} \quad ; \; 0 \le r \le \frac{N}{2}-1 \rightarrow 7$$

$$g(n) = x(n) + x\left(n+\frac{N}{2}\right)$$

splitting the i/p seq g(n) of length $\frac{N}{4}$.

$$= \sum_{n=0}^{\frac{N}{4}-1} g(n)\, W_{\frac{N}{2}}^{rn} + \sum_{n=\frac{N}{4}}^{\frac{N}{2}-1} g(n)\, W_{\frac{N}{2}}^{rn} \rightarrow ⑦$$

substituting $x := n - \frac{N}{4}$ in 2^{nd} summation

$$\therefore n = x + \frac{N}{4}$$

$$X(2r) = \sum_{n=0}^{\frac{N}{4}-1} g(n) W_{\frac{N}{2}}^{rn} + \sum_{x=0}^{\frac{N}{4}-1} g\left(x+\frac{N}{4}\right) W_{\frac{N}{2}}^{r\left(x+\frac{N}{4}\right)}$$

since x is a dummy variable, it can be replaced by n

$$= \sum_{n=0}^{\frac{N}{4}-1} g(n) W_{\frac{N}{2}}^{rn} + \sum_{n=0}^{\frac{N}{4}-1} g\left(n+\frac{N}{4}\right) \cdot W_{\frac{N}{2}}^{rn} W_{\frac{N}{2}}^{r \cdot \frac{N}{4}}$$

$$\cancel{W_{\frac{N}{2}}^{rn}} =$$

$$W_{\frac{N}{2}}^{r \cdot N/4} = W_N^{r \cdot 2 \cdot \frac{N}{4}} = W_N^{r \cdot \frac{N}{2}} , e^{-j\frac{2\pi}{N} \cdot \frac{Nr}{2}} = e^{-j}$$

$$= (-1)^r$$

$$= \sum_{n=0}^{\frac{N}{4}-1} g(n) W_{\frac{N}{2}}^{rn} + \sum_{n=0}^{\frac{N}{4}-1} g\left(n+\frac{N}{4}\right) W_{N/2}^{rn} \cdot (-1)^r$$

$$X(2r) = \sum_{n=0}^{\frac{N}{4}-1} \left[g(n) + (-1)^r g\left(n+\frac{N}{4}\right)\right] W_{\frac{N}{2}}^{rn} \rightarrow (7)$$

substituting $r = 2r$ in eq (7) to get even term

$$X(4r) = \sum_{n=0}^{\frac{N}{4}-1} \left[g(n) + (-1)^{2r} g\left(n+\frac{N}{4}\right)\right] W_{\frac{N}{2}}^{2r \cdot n}$$

$$\left[(-1)^{2r} = 1 \quad \forall r \right.$$

$$\left. W_{\frac{N}{2}}^{2rn} = W_{\frac{N}{4}}^{rn} \right]$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left[g(n) + g\left(n+\frac{N}{4}\right)\right] W_{\frac{N}{4}}^{rn}$$

$$\boxed{X(4r) = \sum_{n=0}^{\frac{N}{4}-1} a(n)\, W_{\frac{N}{4}}^{rn} \quad 0 \le r \le \frac{N}{4}-1} \rightarrow (8)$$

to get odd terms substitute $r = 2r+1$ in eq (7)

$$X(4r+2) = \sum_{n=0}^{\frac{N}{4}-1} \left[g(n) + (-1)^{2r+1} g\left(n+\frac{N}{4}\right)\right] W_{N/2}^{(2r+1)n}$$

$$= \sum_{n=0}^{N/4-1} \left[g(n) + (-1)^{2r+1} g\left(n+\frac{N}{4}\right)\right] W_{N/2}^{2rn} \cdot W_{N/2}^{n}$$

$$(-1)^{2r+1} = -1 \;\forall r \qquad W_{N/2}^{n} = W_N^{2n}$$

$$W_{N/2}^{2rn} = W_{N/4}^{rn}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left[g(n) - g\left(n+\frac{N}{4}\right)\right] W_{\frac{N}{2}}^{n} \cdot W_{\frac{N}{4}}^{rn}$$

$$\rightarrow W_N^{2n}$$

$$\boxed{X(4r+2) = \sum_{n=0}^{\frac{N}{4}-1} b(n) \cdot W_{\frac{N}{4}}^{rn} \quad 0 \le r \le \frac{N}{4}-1} \rightarrow (9)$$

$$\boxed{\begin{aligned} b(n) &= \left[g(n) - g\left(n+\frac{N}{4}\right)\right] W_{\frac{N}{2}}^{n} \\ a_n &= \left[g(n) + g\left(n+\frac{N}{4}\right)\right] \end{aligned}} \rightarrow (10)$$

III^{ly} consider eq (5) $\quad X(4r+1)$ & $X(4r+3)$

$$\boxed{\begin{aligned} c(n) &= \left[h(n) + h\left(n+\frac{N}{4}\right)\right] \\ d(n) &= \left[h(n) - h\left(n+\frac{N}{4}\right)\right] W_{\frac{N}{2}}^{n} \end{aligned}} \rightarrow (11)$$

N = 8

$n = 0 \text{ to } \frac{N}{4} - 1 = 0 \text{ to } \frac{8}{4} - 1$

$= 0 - 1$

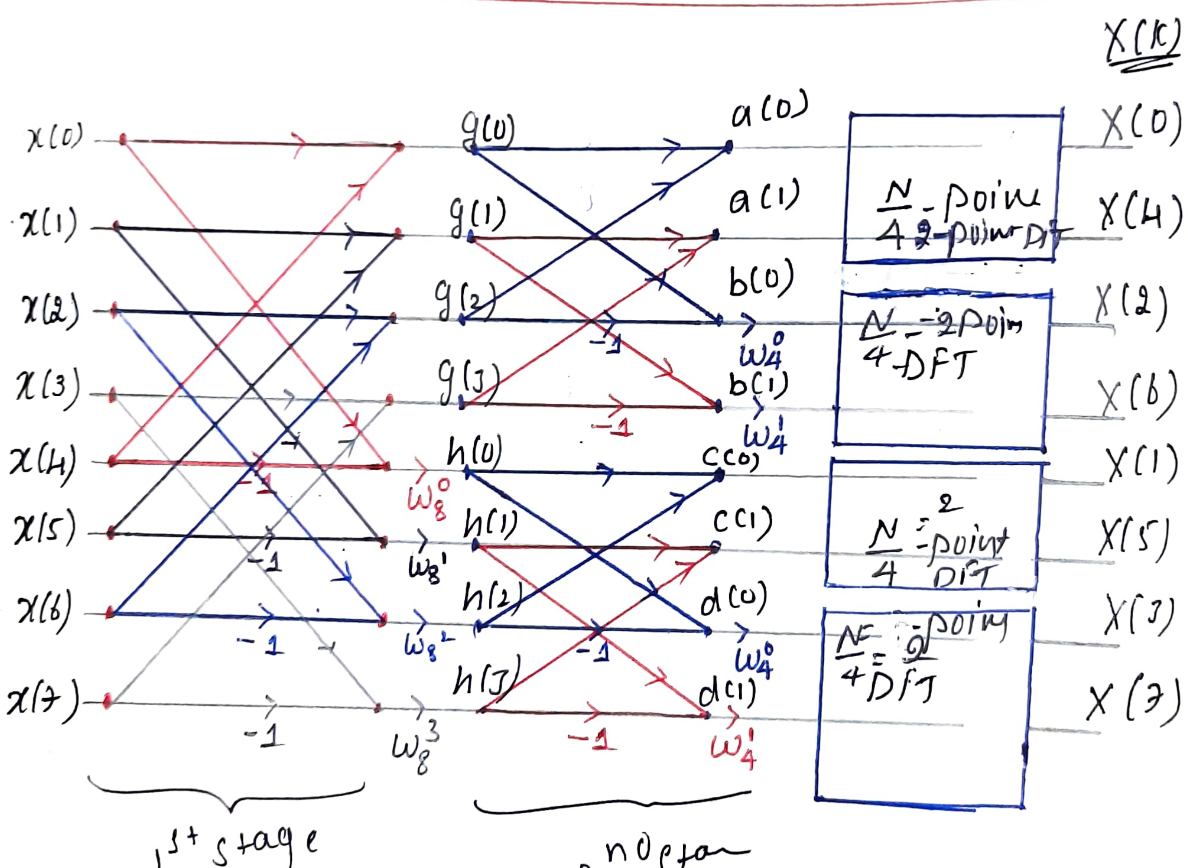
$a(n) = g(n) + g(n+\frac{N}{4}) = g(n) + g(n+2)$

$$\boxed{\begin{aligned} a(0) &= g(0) + g(2) \\ a(1) &= g(1) + g(3) \end{aligned}}$$

$b(n) = [g(n) - g(n+\frac{N}{4})]\, W_{\frac{N}{2}}^{n} = [g(n) - g(n+2)]\, W_4^n$

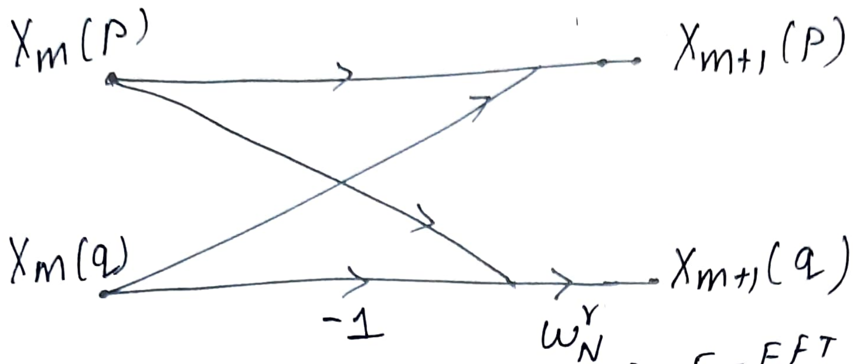
$$\boxed{b(0) = [g(0) - g(2)]\, W_4^0 \quad b(1) = [g(1) - g(3)]\, W_4^1}$$

$$\boxed{\begin{aligned} c(0) &= h(0) + h(2) & c(1) &= h(1) + h(3) \\ d(0) &= [h(0) - h(2)]\, W_4^0 & d(1) &= [h(1) - h(3)]\, W_4^1 \end{aligned}}$$



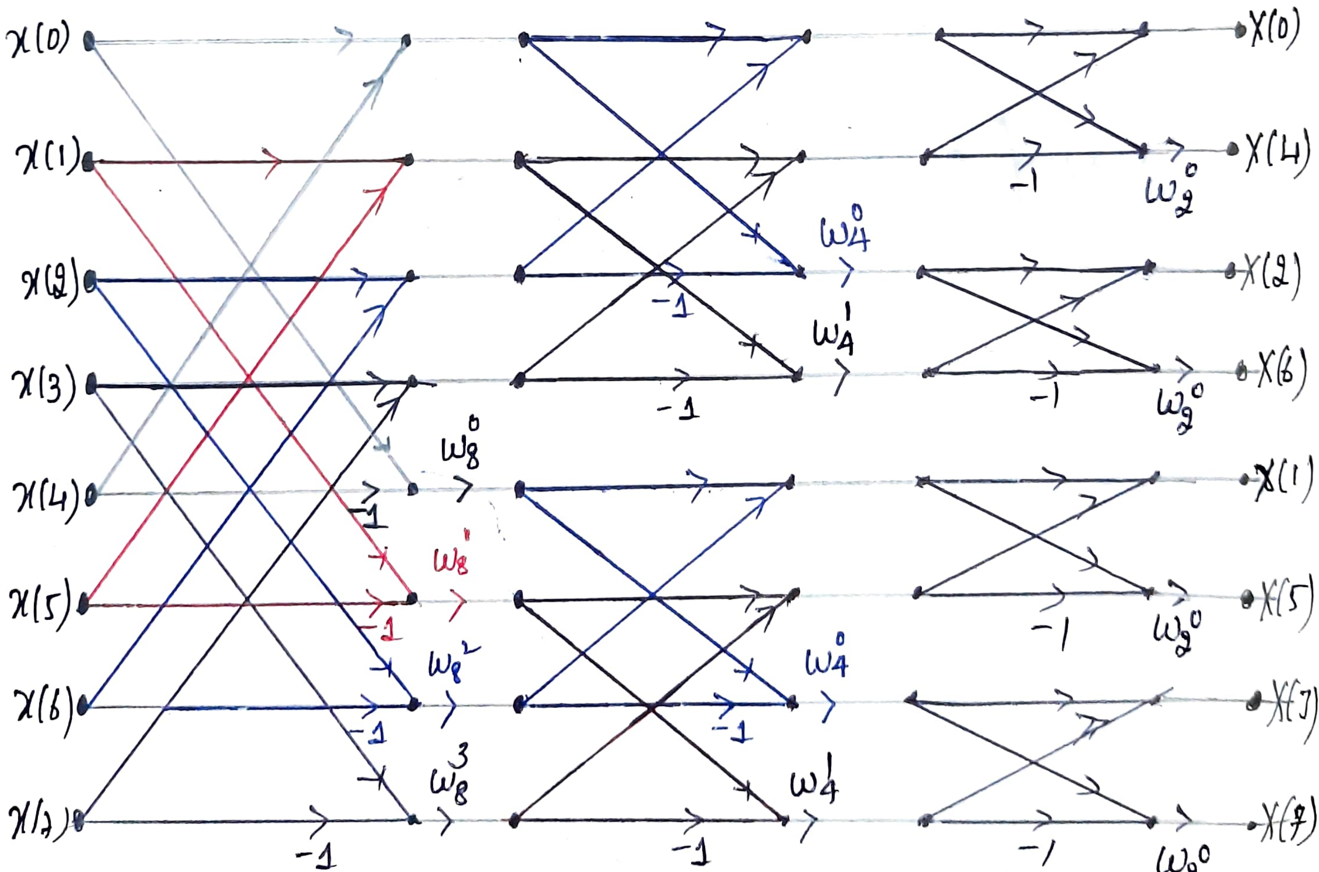
In ~~similar~~ similar way each $\frac{N}{4}$-point DFT is decimated into two $\frac{N}{8}$-point DFT. Continuing this process, down to 2-point DFT, the N-point DFT will be available at the o/p of 2-point transform in bit-reversed order.

The complete process ~~contin~~ consists of $V = \log_2^N$ stages of decimation, where each stage involves $\frac{N}{2}$ butterfly of the type shown below



basic butterfly for DIF-FFT Alg

* The DIF-FFT Alg requires $\frac{N}{2}\log_2^N$ complex x'ions & $N\log_2^N$ complex additions



* We observe that i/p is in normal order where as the o/p is in bit reversed order

* $X_{m+1}(p) = X_m(p) + X_m(q)$

$X_{m+1}(q) = [X_m(p) - X_m(q)]\,W_N^r$

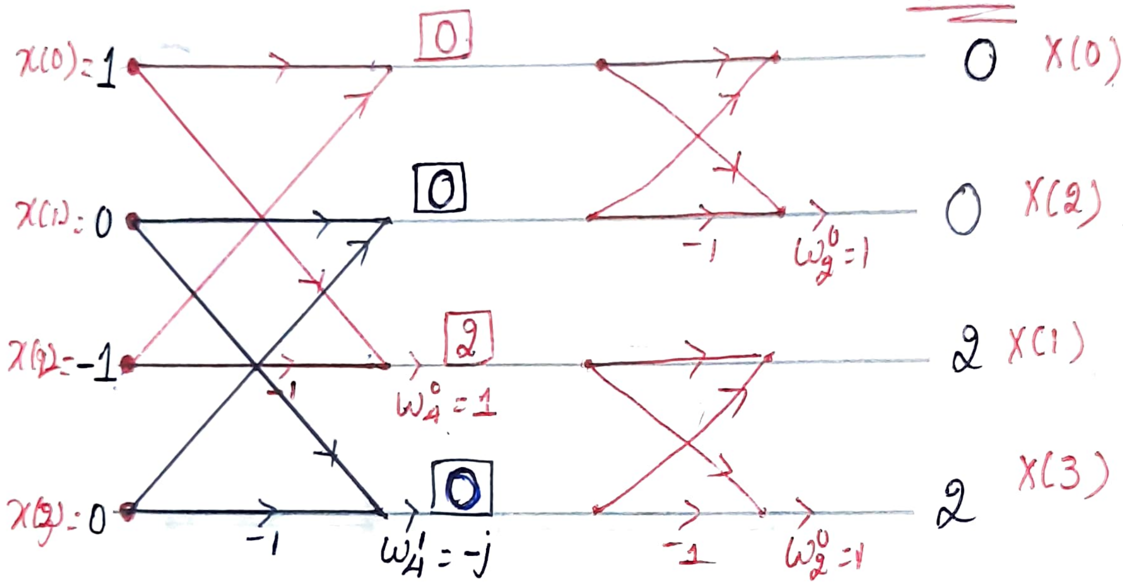
① Compute the DFT of the seq^n $x(n) = \cos n\frac{\pi}{2}$, where $N=4$ using DIF-FFT Alg.

Sol^n:

$$x(n) = \cos n\frac{\pi}{2}$$

$$x(n) = \{1, 0, -1, 0\}$$

The DIF-FFT Alg for $N=4$



$$X(K) = \{0, 2, 0, 2\}$$

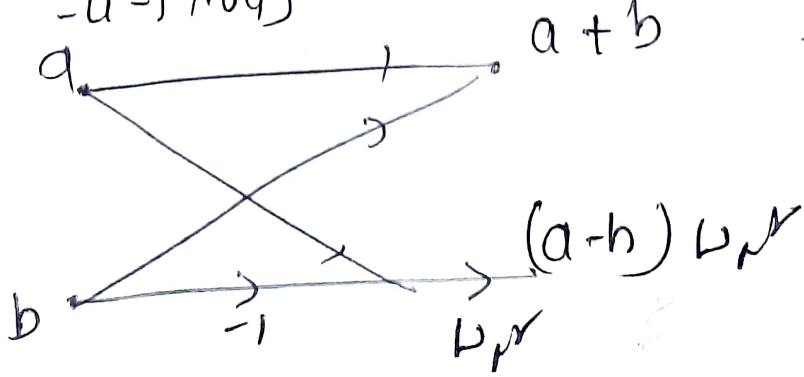
② $x(n) = n+1 \quad 0 \le n \le 7 \quad N=8$

$W_8^0 = 1$, $W_8^1 = 0.707 - j0.707$, $W_8^2 = -j$

$W_8^3 = -0.707 - j0.707$

$x(n) = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$X(K) = \{36, -4+j9.64, -4+4j, -4+j1.64, -4, -4-j1.64, -4-4j, -4-j9.64\}$



Inverse DFT using FFT Algorithm :-

we have

IDFT

$$x(n) \triangleq \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, W_N^{-Kn} \;;\; 0 \le n \le N-1 \quad \text{①}$$

& DFT

$$X(K) \triangleq \sum_{n=0}^{N-1} x(n)\, W_N^{Kn} \;;\; 0 \le K \le N-1 \quad \text{②}$$

by comparing eq ① & ② we find that the only difference bet' 2 eqn's is the factor $\frac{1}{N}$ & sign of power of factor W_N.

$\therefore$ an FFT Alg for computing DFT can be ~~used to~~ converted to an FFT algorithm for computing the IDFT by making the foll changes.

(i) Reverse the direc n of flow-graph

(ii) Change the sign of the power of the factor W_N.

(iii) Replace $x(n)$ by $X(K)$ & vice versa

(iv) multiply the o/p by factor $\frac{1}{N}$

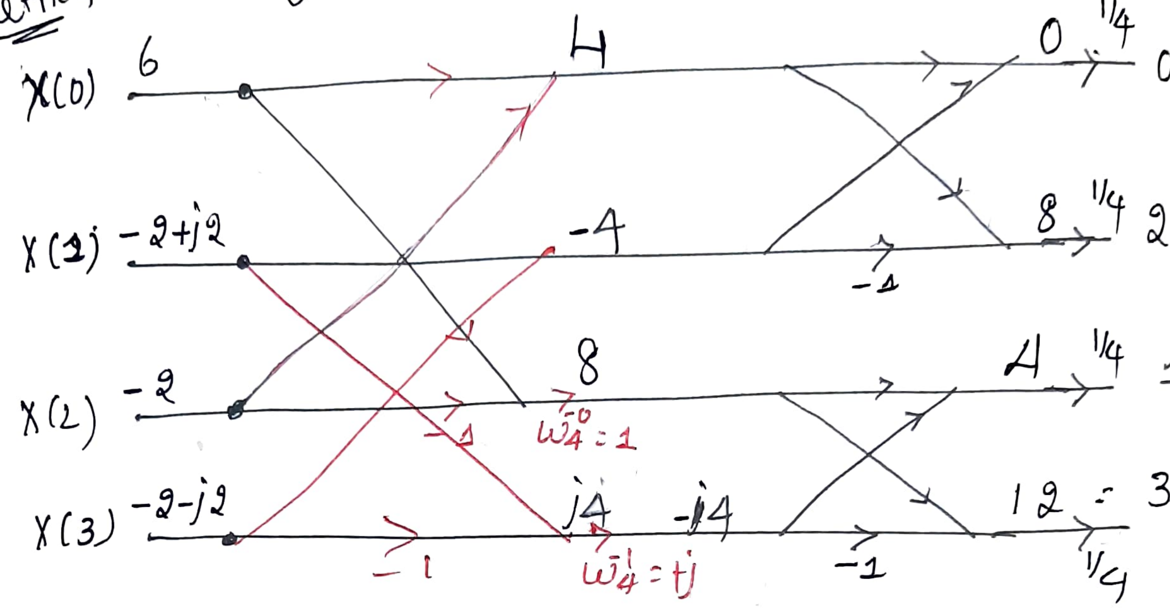
→ I - {6, 8, 10, 12, −4, −2.828 + j2.828, 4j, 2.828 + j2.828}

II - {16, 20, −4, 4j, −4 + 4j, 5.656j, −4, −4j, 5.656j}

III : 36, −4, −4 + 4j, −4 − 4j, −4 + j9.64, −4 − j1.64, −4 + j1.64, −4 − j9.64

Ex ① Find the IDFT of the seq^n given below using DIT-FFT Alg $X(K) = \{6, -2+j2, -2, ...$

I-method U.K & Kumarswamy



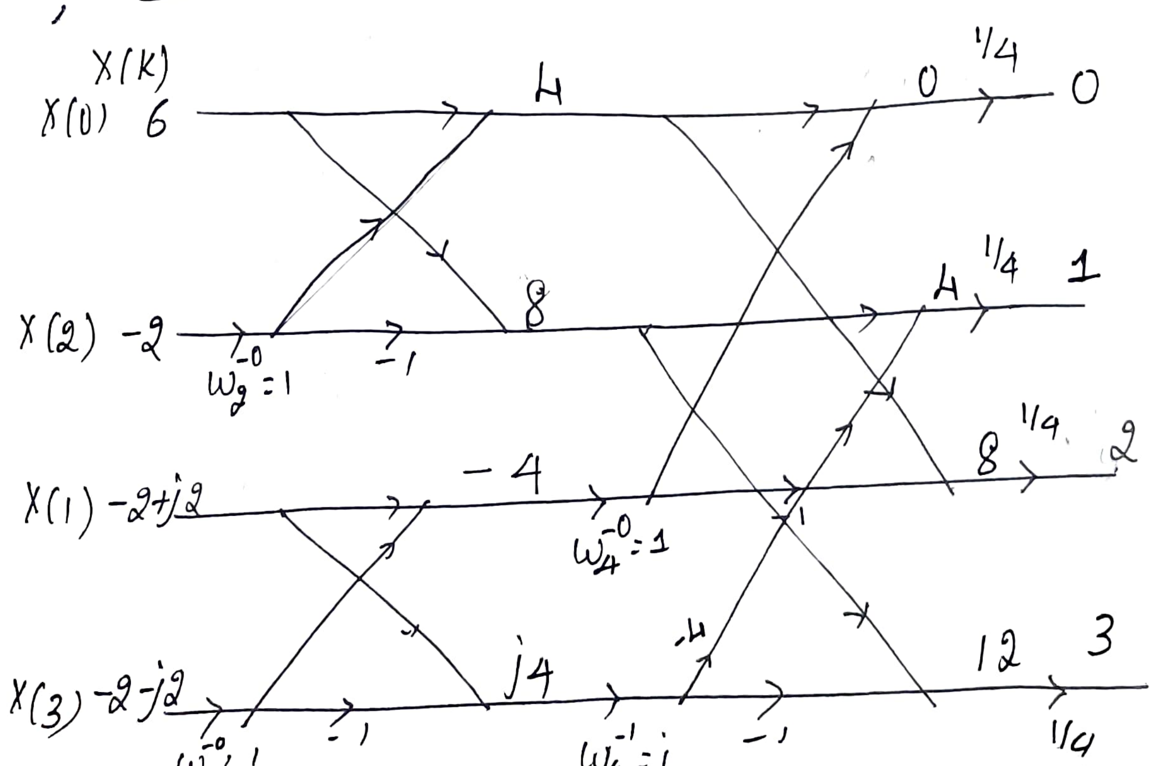
$6 - 2 =$

$-2 + j2 - 2 - j2 = -4$

$-2 + j2 + 2 + j2 = j4(j) = j^2 4$

$x(n) = \{0, 1, 2, 3\}$

II-method G.R



$-2 + j2 - 2 - j2$

$-2 + j2 + 2 + j2$

$x(n) = \{0, 1, 2, 3\}$

(18)

III - method

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, W_N^{-Kn}$$

$$x^*(n) = \frac{1}{N}\sum_{K=0}^{N-1} X^*(K)\, W_N^{Kn}$$

$$N \cdot x^*(n) = \sum_{K=0}^{N-1} X^*(K)\, W_N^{Kn}$$

$$= y(n)$$

$$\therefore x^*(n) = \frac{1}{N}\, y(n)$$

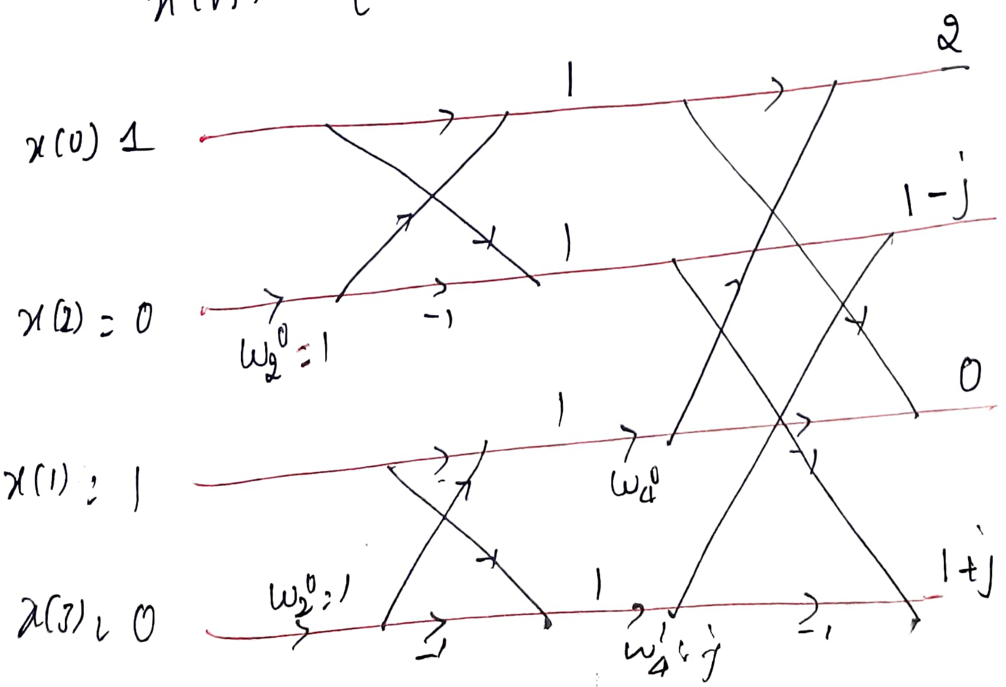
to get IDFT

$$\boxed{x(n) = \frac{1}{N}\, y^*(n)}$$

(2) (a) compute DFT of the seq^n $x(n)$ given below ~~use~~ using DIT-FFT Alg.
$x(n) = \{1, 1, 0, 0\}$.

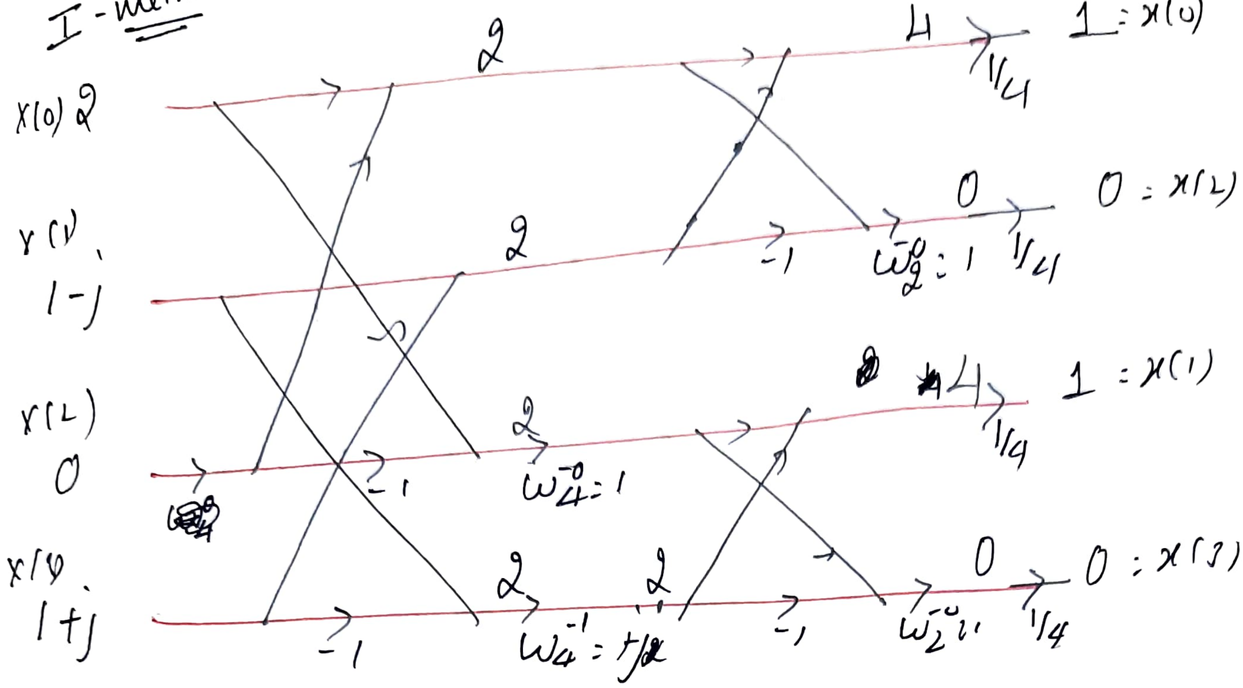
(b) for the $X(K)$ u. DFT found in part (a) find $x(n)$ using DIT-Alg

$x(n) = \{1, 1, 0, 0\}$



$X(K) = \{2,\ 1-j,\ 0,\ 1+j\}$

I - method



$1-j+1+j :$

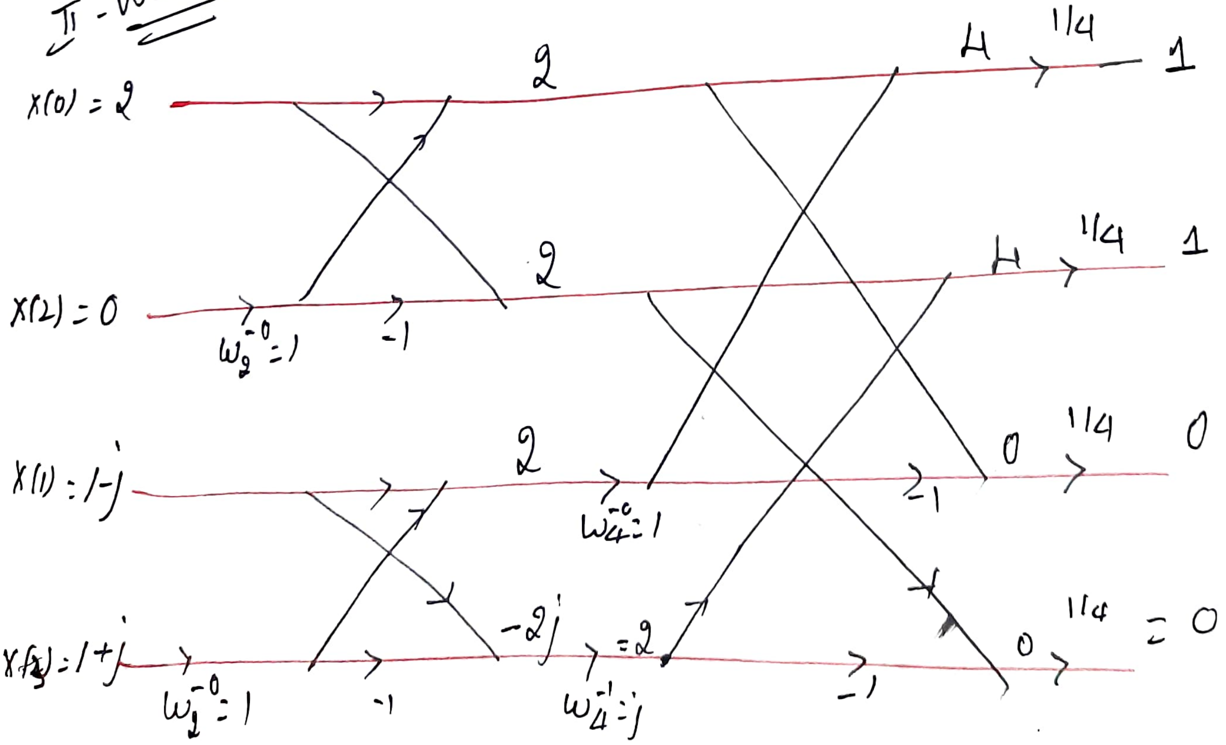
$x(n) : \{1, 1, 0, 0\}$

$1-j-1-j = -2j(+j)$

$= -2j^2$

$= 2$

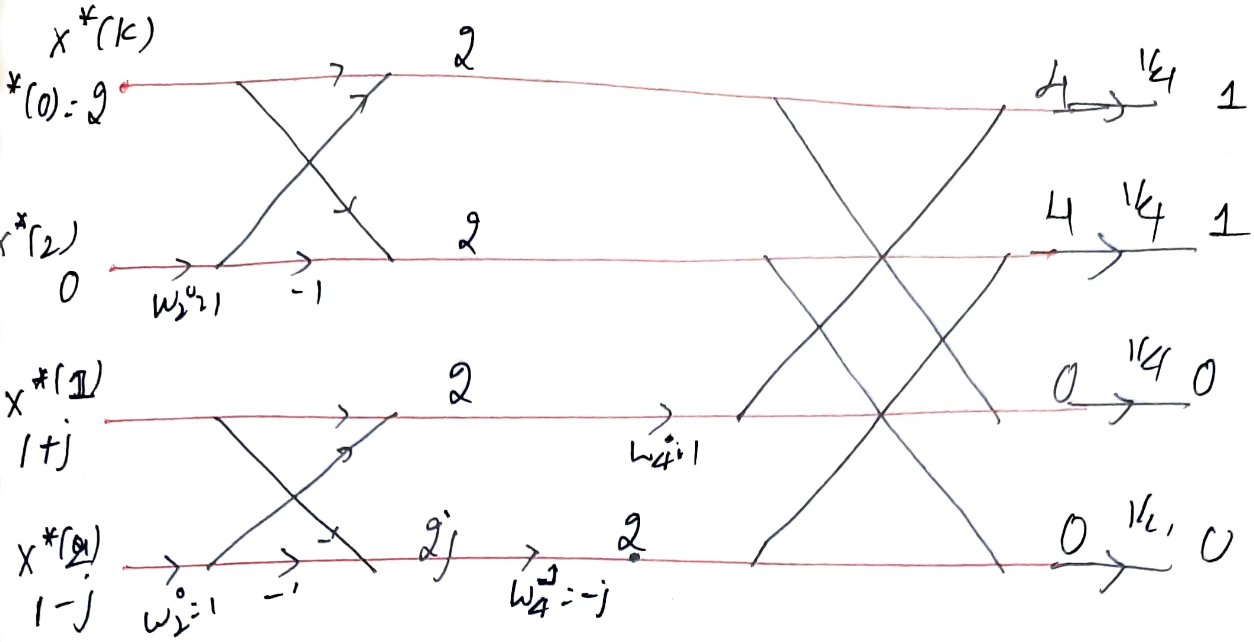
II - method



$1-j+1+j$

$x(n) : \{1, 1, 0, 0\}$

$1-j-1-j$



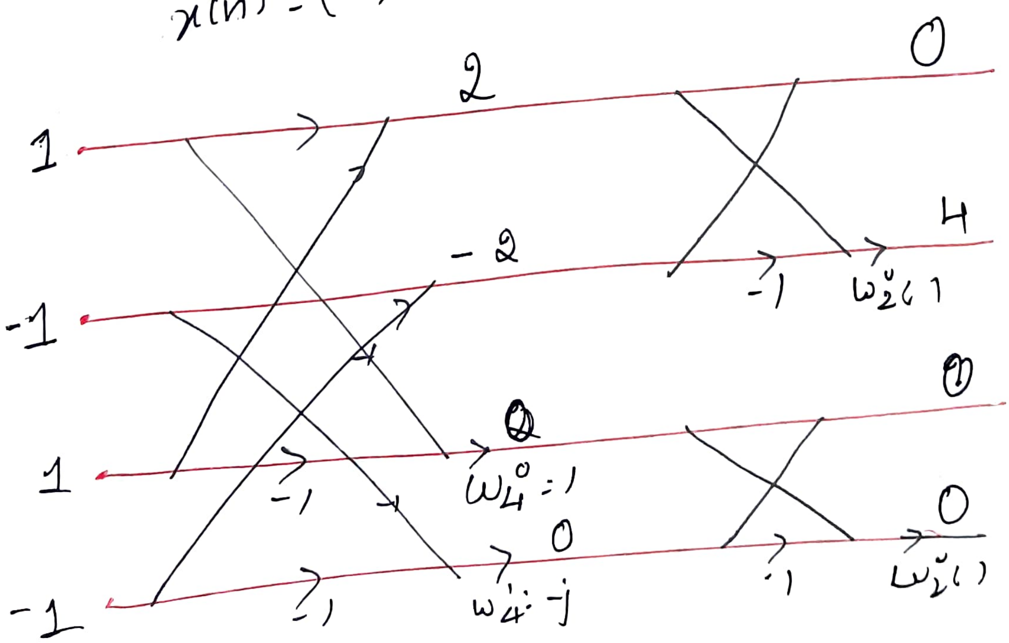
$1+j \quad 1-j$

$2j(-j)$
$= -2j^2$
$= 2$

$1+j \quad -1+j$

(2) Find DFT of seq $x(n) = (-1)^n$ using DIF. obtain IDFT

$x(n) = \{1, -1, 1, -1\}$



$X(k) = \{0, 0, 4, 0\}$

① Find circular convolution of the seqns given below using DIT FFT Alg

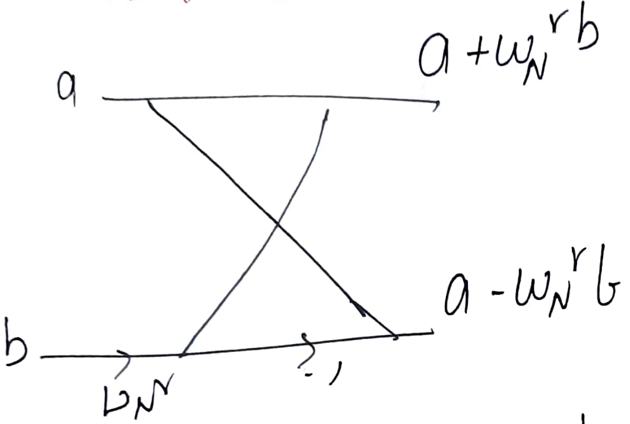
$x_1(n) = \{1, 0, 1, 0\}$

$x_2(n) = \{1, 1, 1, 1\}$

① find circular convolution of the seqns given below

$x_1(n) = \{1, 0, 1, 0\}$

$x_2(n) = \{1, 1, 1, 1\}$



find (1) $X_1(K) = \{2, 0, 2, 0\}$

I - stag. → 2, 0, 0, 0

II → 2, 0, 2, 0

(2) $X_2(K) = \{4, 0, 0, 0\}$

I → 2, 0, 2, 0

II → 4, 0, 0, 0

III $y(n)$

$Y(K) = X_1(K) \cdot X_2(K) = \{8, 0, 0, 0\}$

IDFT using DIT-FFT

→ $y(n) = \{2, 2, 2, 2\}$

I → 8, 8, 0, 0, ~~B~~ $\{2, 2, 2, 2\}$

to find

N-point DFT of 2-real seqn using single N-point DFT

(20)

Let

$g(n) \rightarrow$ real seq^n of length N

$h(n) \rightarrow$ " " " " " "

their DFTs $G(K)$ & $H(K)$

Let $x(n) = g(n) + j\,h(n) \rightarrow (1)$

$g(n) = Re\{x(n)\} = x_R(n) \rightarrow (2)$

$h(n) = Im\{x(n)\} = x_I(n) \rightarrow (3)$

Taking DFT on eq (2) & (3)

using symmetry properties of real seq^n

$$G(K) = \frac{1}{2}\left[X(K) + X^*((-K))_N\right]$$

$$H(K) = \frac{1}{2j}\left[X(K) - X^*((-K))_N\right]$$

$X^*((-K))_N = X^*(N-K)$

determine 4-point DFTs of the 2 real seq^{n}s $g(n) = \{1,2,0,1\}$ $h(n) = \{2,2,1,1\}$ using a single 4-point DFT

$x(n) = g(n) + j\,h(n)$

$x(n) = \{1+j2,\ 2+j2,\ j,\ 1+j\}$

$X(K) = \{4+j6,\ 2,\ -2,\ j2\}$

$X^*(K) = \{4-j6, 2, -2, -j2\}$

$X^*(4-K) = \{4-j6, -j2, -2, 2\}$

$G(K) = \{4, 1-j, -2, 1+j\}$

$H(K) = \{6, 1-j, 0, 1+j\}$

(2) To find 2N-point DFT of a real seq^n using a single N-point DFT

Let $v(n) \rightarrow$ a real seq^n of length 2N
& its DFT $V(K)$.

Let $g(n)$ & $h(n) \rightarrow$ are real seq^n of length N

$$g(n) = v(2n) \qquad 0 \le n \le N-1$$
$$h(n) = v(2n+1)$$

$$x(n) = g(n) + j\,h(n)$$

$$V(K) = \sum_{n=0}^{2N-1} v(n)\, W_{2N}^{Kn}$$

$$\boxed{V(K) = G((K))_N + W_{2N}^{Kn} H((K))_N}$$

$$0 \le K \le 2N-1$$

(21)

find 8-point DFT of the seq n $x(n) = \{1, 2, 2, 2, 0, 1, 1, 1\}$ using a single 4-point DFT

Sol:
$$v(n) = \{1, 2, 2, 2, 0, 1, 1, 1\}$$

$$g(n) = v(2n) = \{1, 2, 0, 1\}$$

$$h(n) = v(2n+1) = \{2, 2, 1, 1\}$$

Let $x(n) = g(n) + j\,h(n)$

$$= \{1+j2,\ 2+j2,\ j,\ 1+j\}$$

$$X(k) = \{4+j6,\ 2,\ -2,\ j2\}$$

$$X^*(k) = \{4-j6,\ 2,\ -2,\ -j2\}$$

$$X^*((-k))_4 = \{4-j6,\ -j2,\ -2,\ 2\}$$

$$G(k) = \{4,\ 1-j,\ -2,\ 1+j\}$$

$$H(k) = \{6,\ 1-j,\ 0,\ 1+j\}$$

$$v(k) = G((k))_N + W_{2N}^{k} H((k))_N$$

$$V(k) = G((k))_4 + W_8^{k} H((k))_4$$

$$\{10,\ 1-j2.4142,\ -2,\ 1-j0.4142,\ -2,\ 1+j0.4142,\ -2,\ 1+j2.4142\}$$

Computation of DFT via filtering methods

(1) Chirp Z-transform:

* WKT DFT of an N-point discrete time signal $x(n)$ is viewed as Z-transform of $x(n)$ evaluated at N equally spaced points on the unit circle

* Here we are interested in evaluating Z-transform on different contours in the complex-plane including the unit circle & spiral contour.

* To derive ~~chirp~~ chirp Z-transform of the seq^n $x(n)$

consider the Z-transform ~~which has~~ for discrete set of values of Z_K,

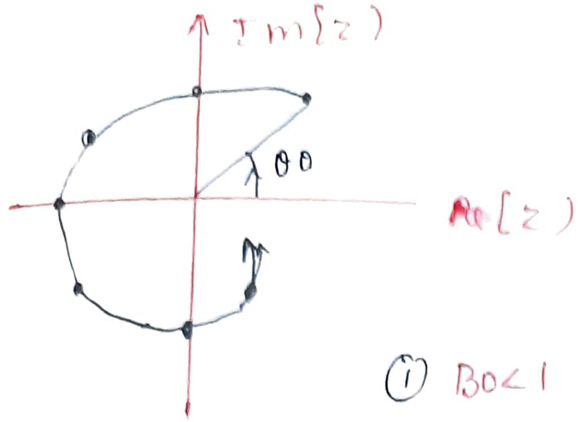
$$X(Z_K) = \sum_{n=0}^{N-1} x(n) Z_K^{-n} \quad ; \quad K = 0, 1, \ldots N-1 \longrightarrow (1)$$

Let $Z_K \rightarrow$ points on the spiral centered about origin.

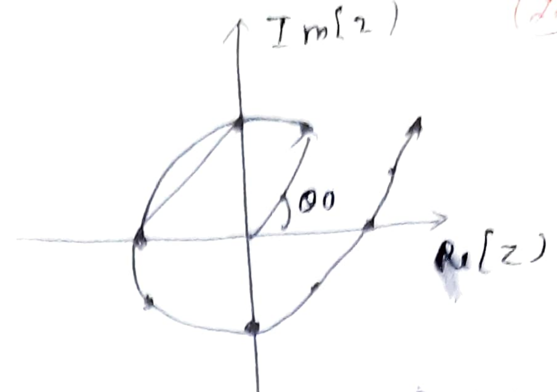
$$Z_K = AB^{-K} \longrightarrow (2)$$

where $A = A_0 e^{j\theta_0} \rightarrow$ defines point at which spiral starts

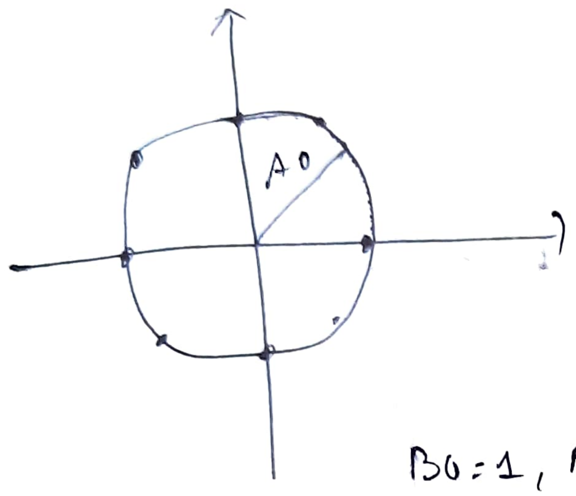
$B = B_0 e^{-j\phi_0} \rightarrow$ defines angular separation b/w points Z_K & determines the rate at which the spiral moves



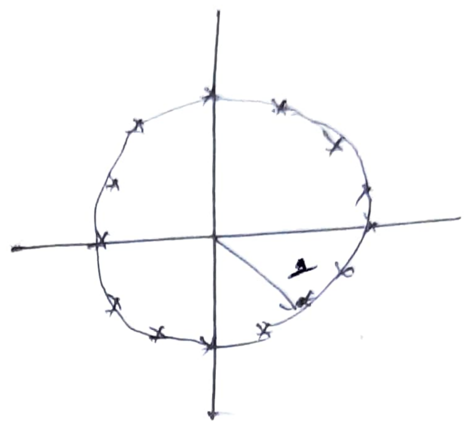
(i) $B_0 < 1$



$B_0 > 1$



$B_0 = 1$, $A_0 < 1$



$A_0 = 1$, $B_0 = 1$

* The above figs shows typical cases for $B_0 < 1$, the seq of points z_k spirals towards the origin

* ~~for $B_0 < 1$~~

* for $B_0 > 1$, the spiral is away from the origin

* for $B_0 = 1$, $A_0 < 1$, the seq of points lies on a circle of radius A_0

* $A_0 = 1$, $B_0 = 1$, corresponds to DFT & seqn of points lies on the unit circle

$\phi = \frac{2\pi}{N}$

Rachin Madan

substituting eq (2) in (1)

$$X(Z_K) = \sum_{n=0}^{N-1} x(n)\ A^{-n} B^{+Kn} \longrightarrow (3)$$

we can express eq (3) in convolution form by consid

$$nK = \frac{1}{2}\left[n^2 + K^2 - \cancel{(K-K)}^2\right] \quad (n-K)^2 \longrightarrow (4)$$

$$X(Z_K) = \sum_{n=0}^{N-1} \underbrace{x(n)\ A^{-n} B^{\frac{n^2}{2}}}\ B^{\frac{K^2}{2}}\ B^{-\frac{\cancel{(K-K)}^2}{2}} \quad B^{-\frac{(n-K)^2}{2}} \longleftarrow (5)$$

Let us define 2 funs g(n) & h(n) as follow

$$g(n) = x(n)\ A^{-n} B^{n^2/2}$$

$$\cancel{h(K) = B^{-K^2/2}}$$

$$h(n) = B^{-n^2/2}$$

$$\therefore g(n) = x(n)\ A^{-n} \frac{1}{B^{-n^2/2}}$$

$$= x(n)\ A^{-n} \cdot \frac{1}{h(n)}$$

$$h(K) = B^{-K^2/2}$$

$$h(n-K) = B^{-(n-K)^2/2}$$

$$\therefore X(Z_K) =$$

$$X(Z_K) = \sum_{n=0}^{N-1} g(n) \cdot \frac{1}{h(K)} \cdot h(n-K)$$

$$= \frac{1}{h(K)} \cdot \sum_{n=0}^{N-1} g(n) \cdot H(n-K)$$

replace n by K & viceversa

$$X(Z_n) = \frac{1}{h(n)} \sum_{K=0}^{N-1} g(K) \cdot H(K-n)$$

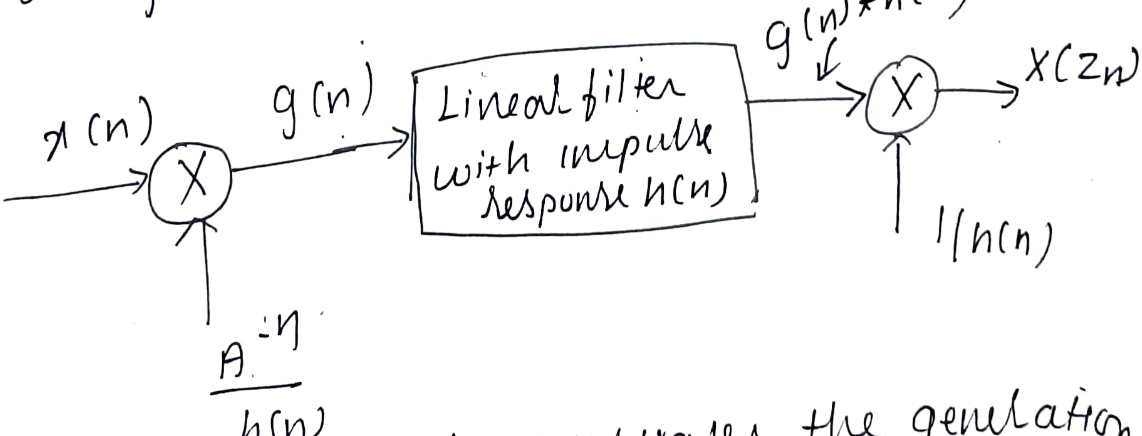
$$x(n) * h(n) = \sum_{K=0}^{N-1} x(k)\, h(n-k)$$

$$= \frac{1}{h(n)} \left[g(n) * h(-n) \right]$$

but $h(-n) = h(n)$ as $h(n)$ is even fun

$$X(Z_n) = \frac{1}{h(n)} \left[g(n) * h(n) \right] \longrightarrow (6)$$

using the above eqn, the foll block dia



The above B.D demonstrates the generation of chirp-Z-transform which is the DFT for the conditions specified in fig (4) if $A_0 = 1$ & $B_0 = 1$ we get

$$h(n) = B^{-n^2/2}$$

$$= \left[B_0 e^{-j\phi_0}\right]^{-n^2/2}$$

$$= e^{+\frac{j\phi_0 n^2}{2}} = \cancel{e^{j\phi_0}} e^{\frac{jn\phi_0 n}{2}} \rightarrow (7)$$

comparing eq (7) with $e^{j\omega n}$ we get

$$\omega = \frac{n\phi_0}{2}$$

~~s~~ eq (7) can be thought of as a complex exponential seqn with a linearly rising freq ω. such a signal is called chirp signals in Radar system hence the name chirp Z-transform

* All the operns illustrated above can be carried out digitally however convolution opern for chrip Z-transform can be implemented by means of Charge transfer devices (CTD) & such devices are avaliable commercially. The CTD imp-lementation appears to be very cheap but not 100% efficient

Goertzel Algorithm :-

* The standard method for spectrum analysis in digital signal processing is the discrete Fourier Transform, implemented using FFT.
* However there are applications that require spectrum analysis only over a subset of N-centre freq's of an N-point DFT
* The popular & efficient technique used is Goertzel Alg.
* Appl'n → dual tone multi freq decoding
  phase-shift keying al
  freq " " modem implementations

Let $x(m)$ is a seq^n of length N
$m = 0 \ldots N-1$

The DFT of $x(m)$

$$X(K) \triangleq \sum_{m=0}^{N-1} x(m) W_N^{Km} \times 1$$

$$= \sum_{m=0}^{N-1} x(m) W_N^{Km} \cdot W_N^{-KN} \text{ (multiply)}$$

$$= \sum_{m=0}^{N-1} x(m) W_N^{-(N-m)K} \longrightarrow ①$$

Since $x(m)$ is defined only bet' $0 \le m \le N-1$ the limits of summation in eq ① can be changed as $-\infty \le m \le \infty$

$$X(K) = \sum_{m=-\infty}^{+\infty} x(m)\, W_N^{-K(N-m)} \longrightarrow (2)$$

Let us define a seq^n

$$y_K(n) = \sum_{m=-\infty}^{\infty} x(m)\, W_N^{-K(n-m)} \longrightarrow (3)$$

Then above eq^n (2) becomes

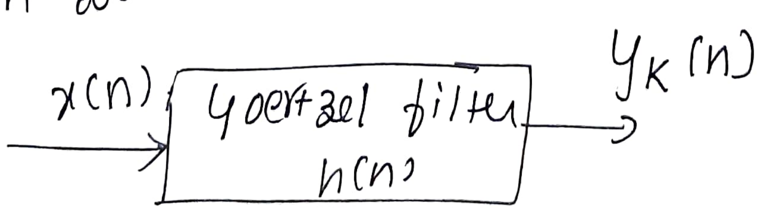
$$X(K) = y_K(n)\Big|_{n=N} \longrightarrow (4)$$

Let $h(n) = W_N^{-nK}$ as a impulse response ~~of~~ a filter known as Goertzels filter
then eq (3) becomes

$$y_K(n) = \sum_{m=-\infty}^{\infty} x(m).\, h(n-m)$$

$$= x(n) * h(n) \longrightarrow (5)$$

eq (5) is ~~rela~~ realized using the B.D shown below



* ~~Let~~ .

Let us now proceed to find the transfer fun ~~of~~ impulse response of Goertzel's filter

$$\frac{Y_k(z)}{X_k(z)} = H_K(z) = Z\{h(n)\}$$

$$= \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(W_N^{-nk}\right) z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(W_N^{-k} z^{-1}\right)^n$$

wkt $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ $|a| < 1$

$$H_K(z) = \frac{Y_K(z)}{X(z)} = \frac{1}{1 - W_N^{-K} z^{-1}}$$

$$Y_K(z)\left[1 - W_N^{-k} z^{-1}\right] = X(z)$$

$$Y_K(z) - W_N^{-K} z^{-1} Y_K(z) = X(z)$$

taking inverse z-transform

$$y_K(n) - W_N^{-K} y_K(n-1) = x(n)$$

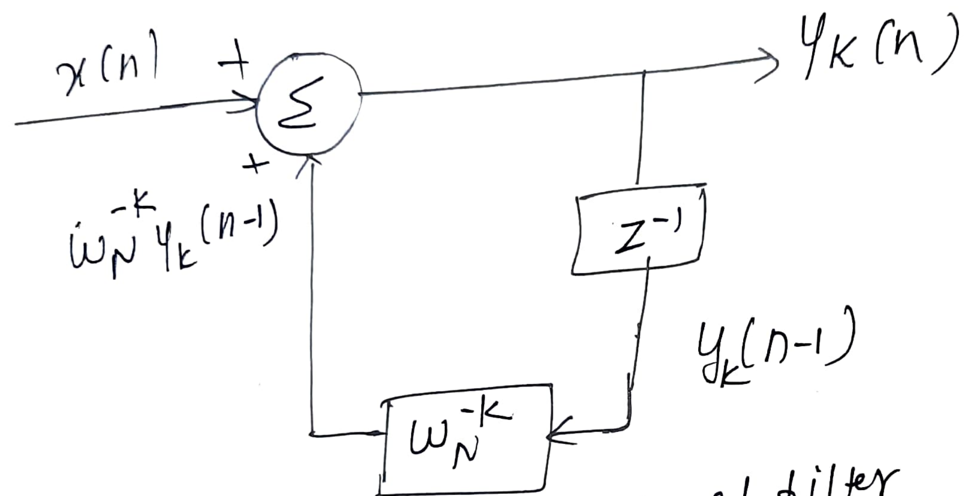
$$\therefore \boxed{y_K(n) = x(n) + W_N^{-K} y_K(n-1)} \rightarrow (6)$$

here delay = 1 hence it is 1st order filter

Assuming $y_K(-1) = 0$ eq (5) is solved recursively to find $y_K(N)$ then from eq (2)

$$X(k) = y_K(N)$$

using eq (7) we can show the B.D of 1st order Goertzel's filter as below



B.D of I order Goertzel filter

the final o/p of 1st order Goertzel filter requires N complex x'ions to compute the o/p at the N^{th} sample.

* The complex x'ion in the above eqn can be reduced by combining complex conjugate poles

$$H_K(z) = \frac{1}{1 - W_N^{-k} z^{-1}}$$

26

$$H_K(z) = \frac{1}{1 - W_N^{-K} z^{-1}} \times \frac{(1 - W_N^{K} z^{-1})}{(1 - W_N^{K} z^{-1})}$$ only conjugate

$$= \frac{1 - W_N^{K} z^{-1}}{1 - W_N^{-K} z^{-1} - W_N^{K} z^{-1} + z^{-2}}$$

$$= \frac{1 - W_N^{K} z^{-1}}{1 - z^{-1}\left[W_N^{-K} + W_N^{K}\right] + z^{-2}}$$

WKT $\left[W_N^{-K} + W_N^{K}\right] = e^{+j\frac{2\pi}{N}K} + e^{-j\frac{2\pi}{N}K}$

$$= 2\cos\frac{2\pi}{N}K$$

$$H_K(z) = \frac{1 - W_N^{K} z^{-1}}{1 - z^{-1}\, 2\cos\frac{2\pi}{N}K + z^{-2}}$$

$$H_K(z) = \underbrace{H_1(z)}_{\text{poles of } H_K(z)} \cdot \underbrace{H_2(z)}_{\text{zero's of } H_K(z)}$$

$$H_K(z) = \frac{1}{1 - 2z^{-1}\cos\frac{2\pi}{N}K + z^{-2}} \left(1 - W_N^{K} z^{-1}\right)$$

$$x(n) \rightarrow \boxed{H_1(z)} \xrightarrow{V_K(n)} \boxed{H_2(z)} \rightarrow Y_K(n)$$

$$H_1(z) = \frac{V_K(Z)}{X(z)} = \frac{1}{1 - 2z^{-1}\cos\frac{2\pi}{N}k + z^{-2}}$$

$$V_K(Z)\left[1 - 2z^{-1}\cos\frac{2\pi}{N}k + z^{-2}\right] = X(z)$$

~~IZT~~

$$v_K(z) - 2z^{-1}v_K(z)\cos\frac{2\pi}{N}k + v_K(z)z^{-2} = X(z)$$

Taking IZT

$$v_K(n) - 2v_K(n-1)\cos\frac{2\pi}{N}k + v_K(n-2) = x(n)$$

$$\boxed{v_k(n) = x(n) + 2v_K(n-1)\cos\frac{2\pi}{N}k - v_K(n-2)}$$

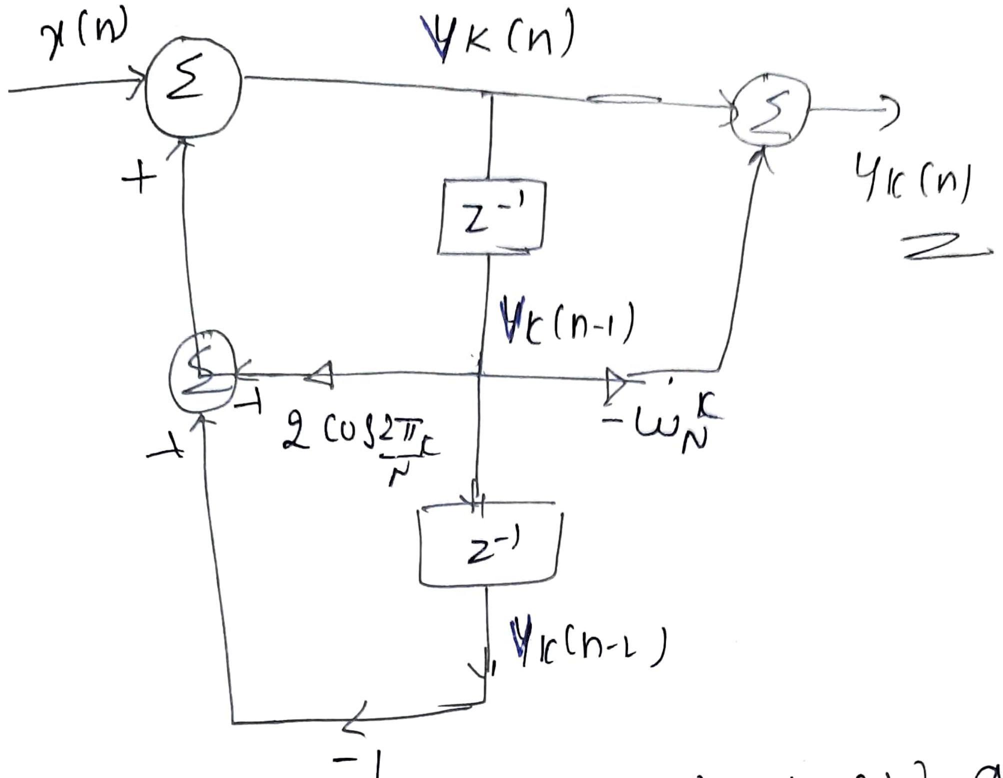
$$H_2(z) = \frac{Y_K(z)}{V_K(z)} = 1 - W_N^k z^{-1}$$

$$Y_K(z) = V_K(z)\left[1 - W_N^k z^{-1}\right]$$

IZT

$$y_K(n) = v_K(n) - W_N^k v_K(n-1)$$

B.D of ~~fr~~ II order filter



Initial conditions $V_K(-1)$ & $V_K(-2)$ are assumed to be 0

The direct form II of II order goertzel filter is shown in above fig

where $X(K) = y_k(n)\big|_{n=N}$

The recursive eqn is ~~iteration~~ iterated for $n = 0$ to N & the final o/p is computed only once at a time $n = N$

each iteration requires one real xions & 2 additions

finally for real seq $x(n)$ requires $(N+1)$ real xion & one complex xion to get $X(K)$ & also $X(N-K)$

when $X(k)$ is to be computed at some value of k, otherwise FFT alg are preferred

① compute $X(2)$ & $X(3)$ given $x(n) = \{\underset{\uparrow}{2}, 0, 2, 0\}$ use Goertzel Alg

soln:

According to Goertzel Alg.

$$X(K) = y_K(N) = y_K(n)\big|_{n=N}$$

$$= y_K(4)$$

Lk

$$y_K(n) = W_N^{-K} y_K(n-1) + x(n)$$

$X(k)$

(i) $\underline{X(2)}$ $K = 2$

$$y_2(n) = W_4^{-2} y_2(n-1) + x(n)$$

$$W_4^{-2} = -1$$

$$= -y_2(n-1) + x(n)$$

initial $y_2(-1) = 0$

$$y_2(0) = -y_2(-1) + x(0) = -0 + 2 = 2$$

$$y_2(1) = -y_2(0) + x(1) = -2 + 0 = -2$$

$$y_2(2) = -y_2(1) + x(2) = 2 + 2 = 4$$

$$y_2(3) = -y_2(2) + x(3) = 0 - (4) = -$$

$$X(2) = y_2(4)$$ 0

) compute $X(2)$ & $X(3)$ given

$x(n) = \{ \underset{\uparrow}{2}, 0, 2, 0 \}$ use Goertzel Al

Sol: According to Goertzel Alg

$$X(K) = y_K(n) \big|_{n=N}$$
$$= y_K(4)$$

) $\underline{X(2)}$:

$K = 2, \quad N = 4$

$$X(2) = y_K(4) = y_2(4)$$

WKT $\quad y_K(n) = x(n) + W_N^{-K}\, y_K(n-1)$

$$y_2(n) = x(n) + W_4^{-2}\, y_2(n-1)$$

$$W_4^{-2} = -1$$

$$y_2(n) = x(n) - y_2(n-1) \longrightarrow (1)$$

initial $y_2(-1) = 0$

$n=0 \quad y_2(0) = x(0) - y_2(-1) = 2 - 0 = \underline{2}$

$n=1 \quad y_2(1) = x(1) - y_2(0) = 0 - 2 = -2$

$n=2 \quad y_2(2) = x(2) - y_2(1) = 2 - (-2) = 4$

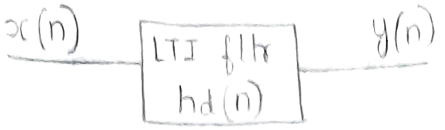
$n=3 \quad y_2(3) = x(3) - y_2(2) = 0 - 4 = -4$

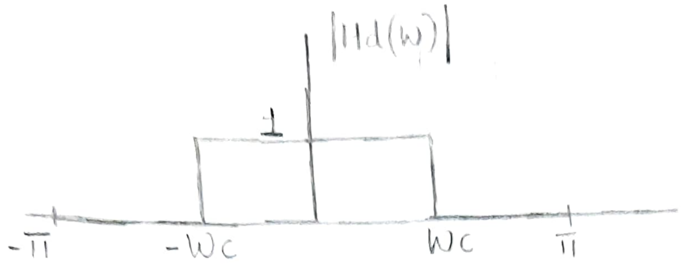
$y_2(4) = x(4) - y_2(3) = 0 - (-4) = 4$

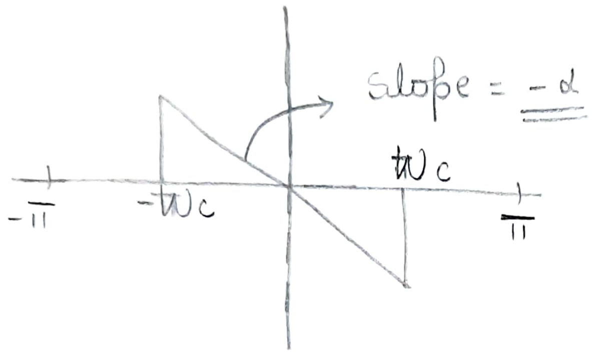
$\cancel{X(2) = y}$

$$\boxed{X(2) = y_2(4) = 4}$$

FINITE IMPULSE RESPONSE FILTER (FIR)







IIR filters were designed to give a desired magnitude response without regards to phase response. In many applicatn, a linear phase is required through out pass band. In order to preserve shape of ~~stop~~ band. signal

Assume desired LPF, frequency response defined by

$$H_d(\omega) = \begin{cases} e^{-j\alpha\omega}, & 0 < |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

$$|H_d(\omega)| = \begin{cases} 1, & 0 < |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

$$\theta(\omega) = \begin{cases} -\omega\alpha, & 0 < |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

Using eqn (2) and (3), mag and phase spectra shown in fig (2) and (3) are drawn.

o/p of filter shown in fig (1) in freqncy domain is

$$Y(\omega) = X(\omega)\, Hd(\omega)$$

Sub eqn (1) in (4), we get

$$Y(\omega) = \begin{cases} e^{-j\alpha\omega} X(\omega), & 0 < |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

Taking IDTF on b.s we get

$$y(n) = \underline{\underline{x(n-\alpha)}}$$

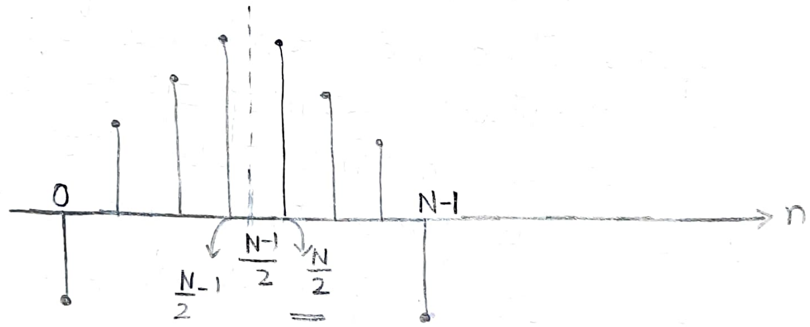
Above eqn means linear phase filter didnot alter the shape of I/P sgl. simply translated it to right by an amout α. If the phase response had not been linear. O/P sgl would be a distorted versn of I/P sgl $x(n)$. It can be shown that causal IIR filter cannot give linear phase. Only special type of FIR filter can give linear phase.

THEORUM FOR LINEAR PHASE:

Statement: let $h(n)$ represent impulse response of a discrete time system. Then necessary and sufficient condtn for existence of linear phase are as follows:

① $h(n)$ must be of finite duratn
② $h(n)$ must be either symmetric or antisymmetric about it midpt
③ For ~~partic~~ practical realizatn, $h(n)$ must be causal seqnce

Case1: Let $h(n)$ be symmetric about its midpt.
Let N be even.



Even symmetry of $h(n)$ about its midpt is explained mathematically by the following discrete eqn

$$h(n) = h(N-1-n) \rightarrow ①$$

WKT $$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

$$H(z) = \sum_{n=0}^{N/2-1} h(n)\, z^{-n} + \sum_{n=N/2}^{N-1} h(n)\, z^{-n}$$

Let us substitute $m = N-1-n$ to the Second summatn on RHS of above eqn. then the above eqn becomes

$$H(z) = \sum_{n=0}^{N/2-1} h(n)\, z^{-n} + \sum_{N/2-1}^{0} h(n)\, z^{-n}$$

$$H(z) = \sum_{n=0}^{N/2-1} h(n) z^{-n} + \sum_{m=N/2-1}^{0} h(N-1-m)\, z^{-(N-1-n)}$$

Since m is a dummy variable, it can be replaced by 'n'. Acc. above eqn becomes

$$H(z) = \sum_{n=0}^{N/2-1} h(n) z^{-n} + \sum_{n=0}^{N/2-1} h(N-1-n)\, z^{-(N-1-n)} \rightarrow (2)$$

Sub eqn (1) in eqn (2), we get

$$H(z) = \sum_{n=0}^{N/2-1} h(n)\left[z^{-n} + z^{-(N-1-n)}\right]$$

Since summatn $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$, FIR system under consideratn is stable. Hence, its freqncy response is obtained by letting $z = e^{jw}$ in $H(z)$

$$H(e^{jw}) = H(w) = \sum_{n=0}^{N/2-1} h(n)\left[e^{-jwn} + e^{-jw(N-1-n)}\right]$$

$$H(e^{jw}) = 2e^{-jw\left(\frac{N-1}{2}\right)} \sum_{n=0}^{N/2-1} h(n)\left[\frac{e^{-jw\left(n-\left(\frac{N-1}{2}\right)\right)} + e^{+jw\left(n-\left(\frac{N-1}{2}\right)\right)}}{2}\right]$$

$$\Rightarrow H(w) = e^{-jw\left(\frac{N-1}{2}\right)} \sum_{n=0}^{N/2-1} 2h(n) \cos w\left[n-\left(\frac{N-1}{2}\right)\right]$$

$$\Rightarrow \boxed{H(w) = e^{-jw\left(\frac{N-1}{2}\right)} \sum_{n=0}^{N/2-1} 2h(n) \cos\left[w\left[\left(\frac{N-1}{2}\right)-n\right]\right] \rightarrow (3)}$$

Comparing eqn (3) with $H(w) = H_r(w)\, e^{j\theta(w)}$ we get

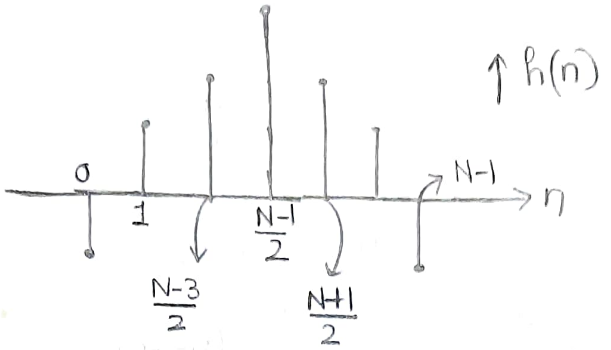
$$H_r(w) = \sum_{n=0}^{N/2-1} 2h(n) \cos\left[w\left[\left(\frac{N-1}{2}\right)-n\right]\right]$$

$$\theta(w) = \begin{cases} -w\left(\frac{N-1}{2}\right) + 0 & ; \text{ if } H_r(w) > 0 \\ -w\left(\frac{N-1}{2}\right) + \pi & ; \text{ if } H_r(w) < 0 \end{cases}$$

Hence phase is linear

Case 2: Let $h(n)$ be even symmetric about it mid pt and N be odd.

$\frac{N-1}{2} - 1$



Even symmetry of $h(n)$ can be explained mathematically by following discrete relatn

$$h(n) = h(N-1-n)$$

WKT

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$2N-2-N-1$ $\frac{N-1}{2}$

$$H(z) = \sum_{n=0}^{\frac{N-3}{2}} h(n) z^{-n} + h\left(\frac{N-1}{2}\right) z^{-\left(\frac{N-1}{2}\right)} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) z^{-n}$$

letting $m = N-1-n$ in the 3rd term on RHS of above eqn, we get

$$H(z) = \sum_{n=0}^{N-3/2} h(n) z^{-n} + h\left(\frac{N-1}{2}\right) z^{-\left(\frac{N-1}{2}\right)} + \sum_{m=\frac{N}{2}-1}^{0} h(N-1-m) z^{-(N-1-m)}$$

Since m is a dummy variable, it can be replaced by 'n'. Acc, above eqn becomes

$$H(z) = h\left(\frac{N-1}{2}\right) z^{-\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n) z^{-n} + \sum_{m=0}^{\frac{N}{2}-1} h(N-1-n) z^{-(N-1-n)}$$

Sub eqn ① in eqn ② and then combing two summatn, we get

$$H(z) = h\left(\frac{N-1}{2}\right) z^{-\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n)\left[z^{-n} + z^{-(N-1-n)}\right]$$

Since impulse response is absolutely summable FIR system under consideration is stable, Hence its impulse response is found by letting $z = e^{j\Omega}$ in $H(z)$

$$H(e^{j\omega}) = H(\omega) = h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{(N-3)/2} h(n)\left\{e^{-j\omega n} + e^{-j\omega(N-1-n)}\right\}$$

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} 2h(n)\left[\frac{e^{-j\omega\left(n-\left(\frac{N-1}{2}\right)\right)} + e^{+j\omega\left(n-\left(\frac{N-1}{2}\right)\right)}}{2}\right]\right]$$

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} 2h(n)\cos\omega\left(n-\left(\frac{N-1}{2}\right)\right)\right]$$

Since $\cos(-\theta) = \cos\theta$ above eqn can be written as

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} 2h(n)\cos\omega\left(\frac{N-1}{2} - n\right)\right] \rightarrow (3)$$

Comparing eqn (3) with

$H(\omega) = H_r(\omega)\, e^{j\theta(\omega)}$, we get

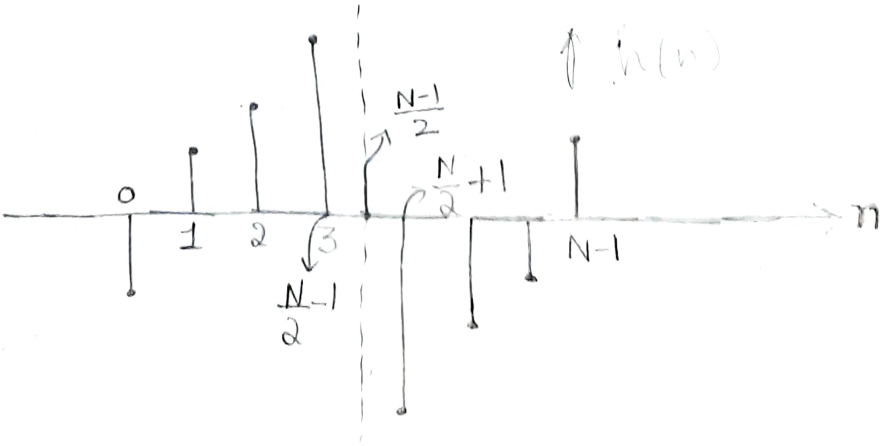
$$\boxed{H_r(\omega) = h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} 2h(n)\cos\omega\left(\frac{N-1}{2} - n\right)}$$

and

$$\theta(\omega) = \begin{cases} -\omega\left(\frac{N-1}{2}\right) + 0, & \text{if } H_r(\omega) > 0 \\ -\omega\left(\frac{N-1}{2}\right) + \pi, & \text{if } H_r(\omega) < 0 \end{cases}$$

hence the phase is linear

<u>Case 3</u>: Let $h(n)$ be the antisymmetry about its midpt and N be even



Odd symmetry of $h(n)$ about its midpt is explained mathematically by the discrete relatn given below

$$h(n) = -h(N-1-n) \rightarrow (1)$$

WKT, $H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$

$$H(z) = \sum_{n=0}^{\frac{N}{2}-1} h(n) z^{-n} + \sum_{n=\frac{N}{2}}^{N-1} h(n) z^{-n}$$

Letting $m = N-1-n$ in second summatn on RHS of above eqn gives

$$H(z) = \sum_{n=0}^{\frac{N}{2}-1} h(n) z^{-n} + \sum_{m=\frac{N}{2}-1}^{0} h(N-1-m) z^{-(N-1-m)}$$

Since m is some dummy variable, it can be replaced by 'n'. Acc above eqn becomes

$$H(z) = \sum_{n=0}^{(N/2)-1} h(n) z^{-n} + \sum_{n=0}^{(N/2)-1} h(N-1-n) z^{-(N-1-n)} \rightarrow (2)$$

Sub eqn (1) in eqn (2) and then combining two summatn, we get

$$H(z) = \sum_{n=0}^{N/2-1} h(n) \left\{ z^{-n} - z^{-(N-1-n)} \right\}$$

Since the system under consideratn is stable $\left(\sum_{n=-\infty}^{\infty} |h(n)| < \infty\right)$, freqncy response is obtained by letting $z = e^{jw}$ in $H(z)$

$$H(e^{jw}) = H(w) = \sum_{n=0}^{N/2-1} h(n) \left\{ e^{-jwn} - e^{-jw(N-1-n)} \right\}$$

$$H(e^{jw}) = H(w) = e^{-jw\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\frac{N}{2}-1} (-2j) h(n) \left\{ \frac{e^{-jw\left(n-\left(\frac{N-1}{2}\right)\right)} - e^{+j\left(w\left(n-\left(\frac{N-1}{2}\right)\right)\right)}}{(-2j)} \right\}$$

$$H(w) = e^{-jw\left(\frac{N-1}{2}\right)} \sum_{n=0}^{N/2-1} \left(-2e^{j\frac{\pi}{2}}\right) h(n) \sin w\left(n-\left(\frac{N-1}{2}\right)\right)$$

Since $\sin(-\theta) = -\sin\theta$ we get

$$H(w) = e^{-jw\left(\frac{N-1}{2}\right)} \sum_{n=0}^{N/2-1} 2e^{j\frac{\pi}{2}} h(n) \sin\left(\frac{N-1}{2} - n\right)$$

$$H(w) = e^{-jw\left(\frac{N-1}{2}\right) + j\frac{\pi}{2}} \sum_{n=0}^{N/2-1} 2h(n) \sin\left(\frac{N-1}{2} - n\right) \rightarrow (3)$$

Comparing eqn (3) with

$H(w) = H_r(w) e^{j\theta(w)}$, we get

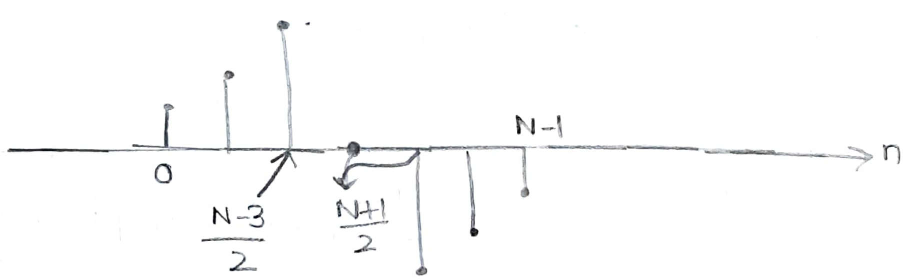
$$H_r(w) = \cancel{e^{-jw\left(\frac{N-1}{2}\right)+j\frac{\pi}{2}}} \sum_{n=0}^{\frac{N}{2}-1} 2h(n) \sin\left(\frac{N-1}{2} - n\right)$$

and

$$\theta(w) = \begin{cases} -w\left(\frac{N-1}{2}\right) + \frac{\pi}{2} + 0, & \text{if } H_r(w) > 0 \\ -w\left(\frac{N-1}{2}\right) + \frac{\pi}{2} + \pi, & \text{if } H_r(w) < 0 \end{cases}$$

hence the phase is linear

<u>Case 4</u> : Let $h(n)$ be odd symmetric about its mid pt and N be odd.



Odd symmetry of $h(n)$ is explained mathematically by the foll. discreate relatn

$$h\left(\frac{N-1}{2}\right) = 0 \rightarrow (1) \text{ and } h(n) = -h(N-1-n) \rightarrow (2)$$

WKT

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$H(z) = \sum_{n=0}^{N-3/2} h(n) z^{-n} + h\left(\frac{N-1}{2}\right) z^{-\left(\frac{N-1}{2}\right)} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) z^{-n}$$

Let $m = N-1-n$ in second summatn

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n)\, z^{-n} + \sum_{m=0}^{\frac{N-1}{2}} h(N-1-m)\, z^{-(N-1-m)}$$

Since m is dummy variable, replace m by n

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n)\, z^{-n} + \sum_{m=0}^{\frac{N}{2}-1} h(N-1-m)\, z^{-(N-1-n)}$$

Sub eqn (1) in eqn (2), and combining the eqn, we get

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[z^{-n} - z^{-(N-1-n)} \right]$$

Since impulse response is absolute summable FIR system is stable, $\left[\sum_{n=-\infty}^{\infty} h(n) < \infty\right]$ Thus its frequency response is given by $H(z)$ replace z by e^{jw}

$$H(e^{jw}) = H(w) = \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[e^{-jwn} - e^{-jw(N-1-n)} \right]$$

$$H(w) = e^{-jw\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\frac{N-3}{2}} h(n)\,(-2j) \left[\frac{e^{-jw\left(n-\left(\frac{N-1}{2}\right)\right)} - e^{+jw\left(n-\left(\frac{N-1}{2}\right)\right)}}{(-2j)} \right]$$

$$H(w) = e^{-jw\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\frac{N-3}{2}} -2e^{j\frac{\pi}{2}}\, h(n) \sin\left(n-\left(\frac{N-1}{2}\right)\right)$$

$$H(w) = e^{-jw\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\frac{N-3}{2}} 2e^{j\frac{\pi}{2}}\, h(n) \sin\left(\frac{N-1}{2} - n\right)$$

$$H_r(w) = \sum_{n=0}^{\frac{N-3}{2}} 2h(n) \sin\left(\frac{N-1}{2} - n\right)$$

$\frac{n-1}{2} = 3$

$n = 6+1$

$$\theta(w) = \begin{cases} -w\frac{(N-1)}{2} + \frac{\pi}{2}, & \text{if } H_r(w) > 0 \\ -w\left(\frac{N-1}{2}\right) + \frac{\pi}{2} + \pi, & \text{if } H_r(w) < 0 \end{cases}$$

phase is linear

☆ Determine unit impulse response $h(n)$ of a linear phase FIR filter · with length $N=4$

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Given $H_r(0)=1$ and $H_r(\pi/2)=1/4$

S: Let us assume $h(n)$ to be even symmetric about its mid pt. Then we have

$$H_r(jw) = \sum_{n=0}^{N/2-1} 2h(n)\cos w\left(\left(\frac{N-1}{2}\right)-n\right)$$

$$N=4$$

$$H_r(jw) = \sum_{n=0}^{1} 2h(n)\cos w(1.5-n)$$

$$H_r(w) = 2h(0)\cos w(1.5) + 2h(1)\cos(0.5)w$$

① $H_r(0)=1 \Leftarrow 1 = 2h(0) + 2h(1)$

② $H_r\left(\frac{\pi}{2}\right)=\frac{1}{4} \Leftarrow \frac{1}{4} = 2h(0)\cos\left(1.5\times\frac{\pi}{2}\right)+2h(1)\cos\left(0.5\,\frac{\pi}{2}\right)$

$$h(0) = 0.1616$$

$$h(1) = 0.3384$$

Since $h(n)$ is even symmetric about its mid pt, foll condtn must be true

$$h(N-1-n) = h(n)$$

$$h(2) = 0.3384$$

$$h(3) = 0.1616$$

$$h(n) = \{0.1616,\ 0.3384,\ 0.3384,\ 0.1616\}$$

☆ The freqency response of an FIR filter is given by $H(w) = e^{-3jw}\{1.2+0.6\cos 3w+0.8\cos 2w + 0.4\cos w\}$ determine Co-eff of FIR filter?

S: here $\frac{N-1}{2} = 3 \Rightarrow \underline{N = 7}$

Since $H(w)$ has cosine terms it implies $h(n)$ is even symmetric about it mid pt. Also N is odd.

Hence, $H(w) = e^{-jw\left(\frac{N-1}{2}\right)} \left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{(N-3)/2} 2h(n) \cos\left[w\left(\frac{N-1}{2} - n\right)\right]\right]$

$\underline{N = 7}$

$$H(w) = e^{-j3w}\left[h(3) + \sum_{n=0}^{2} 2h(n) \cos w(3-n)\right]$$

$$\Rightarrow \quad = e^{-3jw}\left[h(3) + 2h(0)\cos 3w + 2h(1)\cos 2w + 2h(2)\cos w\right] \quad \text{(2)}$$

$\Rightarrow$ Comparing eqn (1) and eqn (2), we get

$h(3) = 1.2$

$h(0) = 0.3$

$h(1) = 0.4$

$h(2) = 0.2$

Since $h(n)$ is even symmetry a

$h(n) = h(N-1-n)$

$\Rightarrow h(n) = h(6-n)$

thus $h(4) = 0.2 = h(2)$

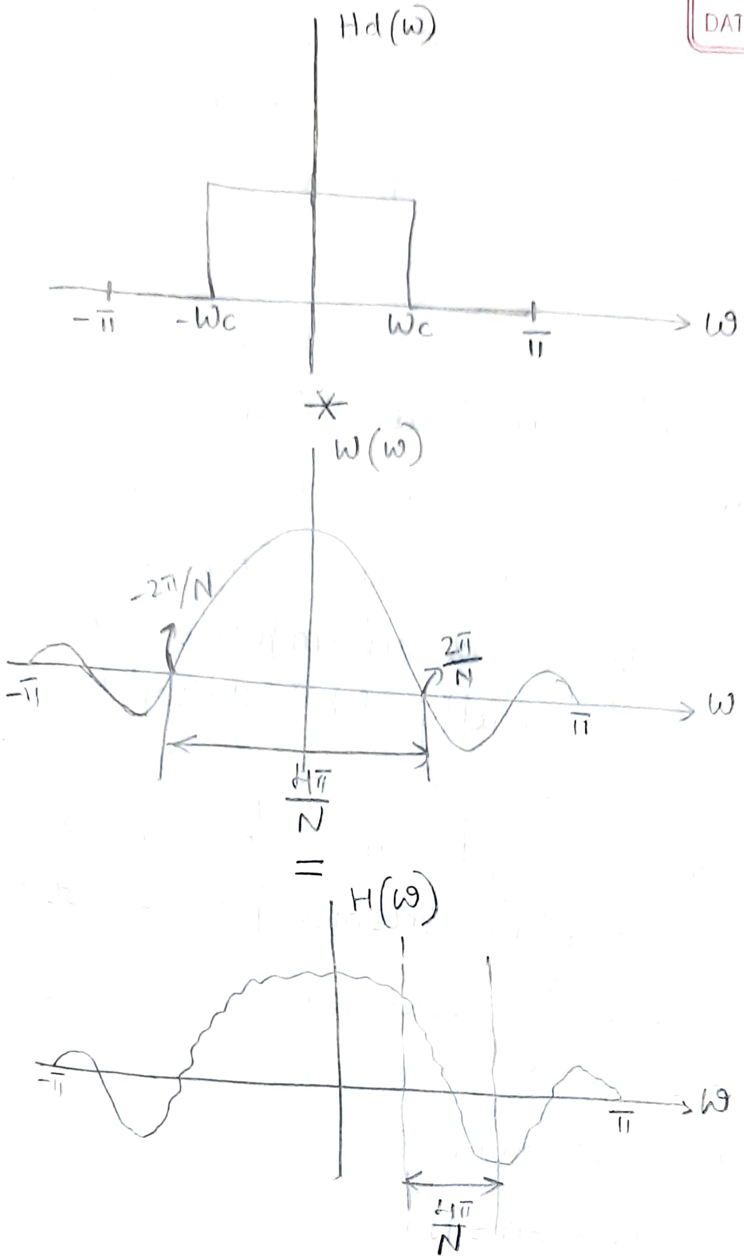
$h(5) = 0.4 = h(1)$

$h(6) = 0.3 = h(0)$

thus

$$h(n) = \{\underset{\uparrow}{0.3}, 0.4, 0.2, 1.2, 0.2, 0.4, 0.3\}$$

FIR FILTER DESIGN USING WINDOWS:



The Easiest way to design an FIR filter is to truncate impulse response of an IIR filter. Let hd(n) represent impulse response of a desired low-p filter [IIR] and h(n) the impulse response of an FIR filter then

$$h(n) = hd(n)\, w(n) \rightarrow (1)$$

$$w(n) = \begin{cases} 1, & N_1 \le 0 \le N_2 \\ 0, & \text{otherwise} \end{cases}$$

W(n) defined above is called rectangular window. Taking DTFT on b.s of eqn (1) we get

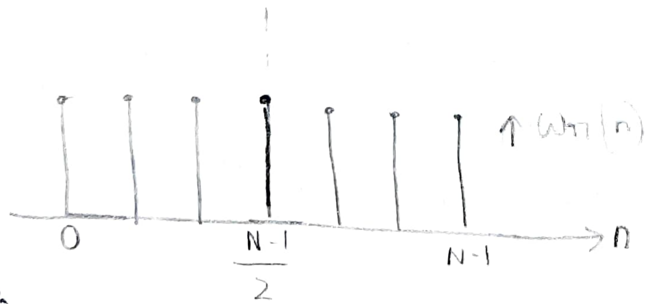
$$H(\omega) = Hd(\omega) * W(\omega) \rightarrow (2)$$

Let us now assume Hd(w) represent freqncy response of ideal low pass filter W(w) freqncy response of a causal rectangular window that starts at n = 0 and ends at n = N-1. Then the freqncy response of FIR filter is as shown in fig (3).

The freqncy response of FIR filter shown in fig (3) is the smeared versn of Hd(w) shown in fig (1). In general, wider the main lobe W(w) more will be the smearing and vice-versa. In precise, if W(w) is an impulse functn, then H(w) will look exactly like Hd(w). Hence, in practice, we make N large enough so that smearing is minimised and yet small enough so that practical implimentatn becomes posible. Some of the most commonly used windows are described below:

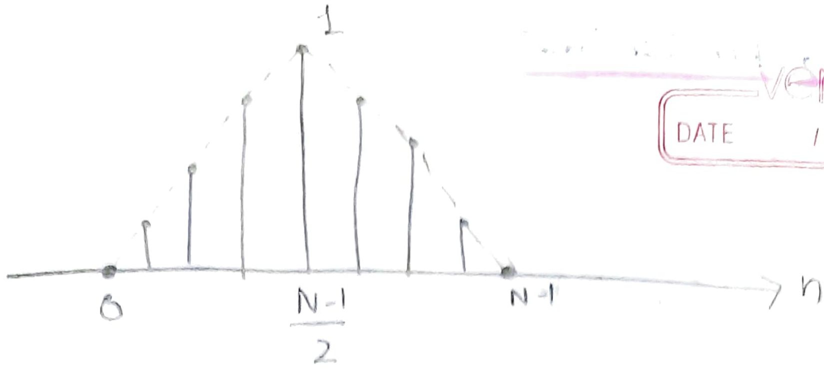
(1) Rectangular window

$$W_R(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$



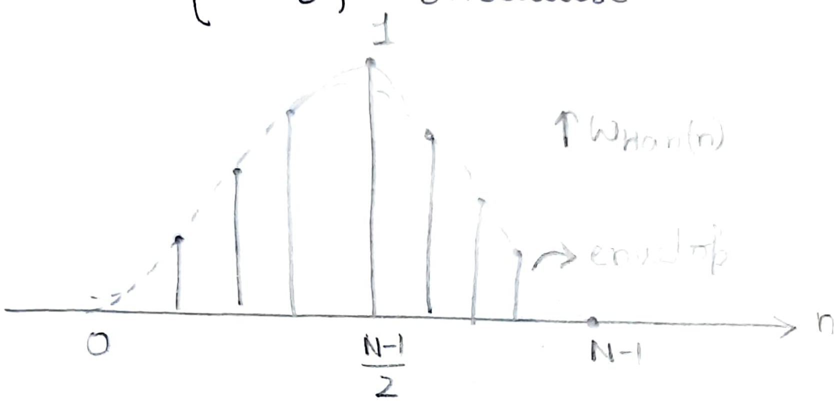
(b) Bartlet window

$$W_R(n) = \begin{cases} 1 - \dfrac{2\left(n - \left(\frac{N-1}{2}\right)\right)}{N-1} & ; 0 \le n \le N-1 \\ 0 & ; \text{o.w} \end{cases}$$



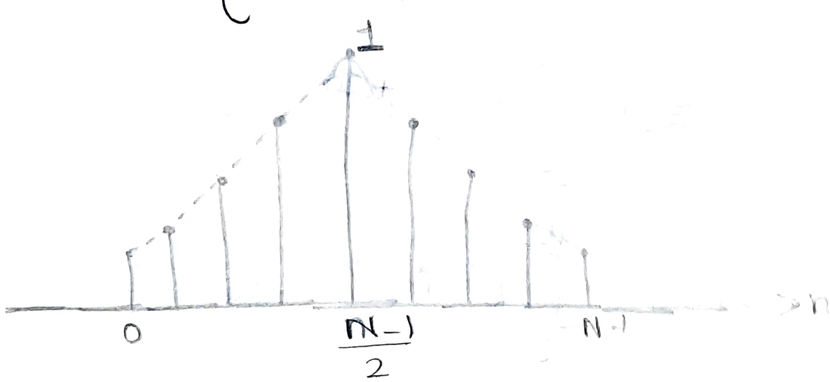
3) Hanning window

$$W_{Han}(n) = \begin{cases} \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{(N-1)}\right)\right], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$



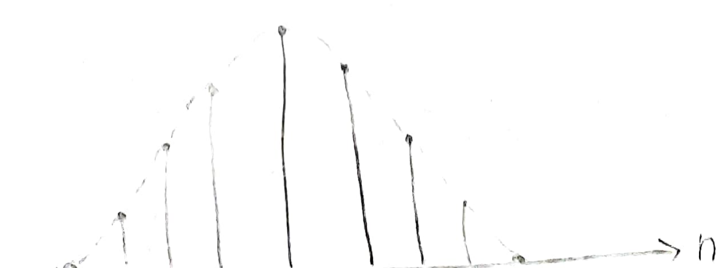
4) Hamming window

$$W_{Ham}(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{(N-1)}\right), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$



5) Blackman window

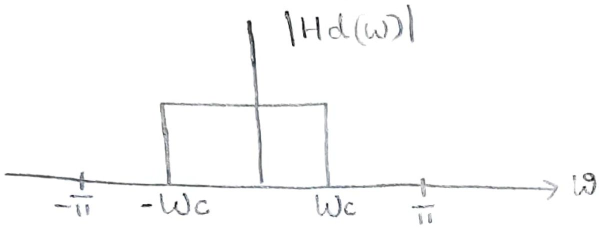
$$W_{Bl}(n) = \begin{cases} 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right); & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

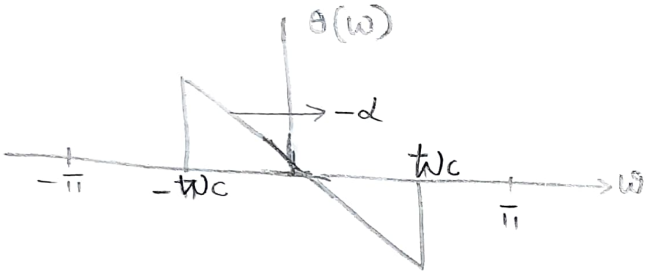


DESIGN PROCEDURE:

Let $Hd(\omega)$ represent frequency response of an ideal low pass filter (IIR). Such a filter has the following mathematical descriptn.

$$Hd(\omega) = \begin{cases} e^{-j\omega\alpha} ; & 0 < |\omega| < \omega_c \\ 0 ; & \omega_c < |\omega| < \pi \end{cases}$$





Impulse response, $hd(n)$ of the ideal low pass filter described above is found as IDFT of $Hd(\omega)$.

$$hd(n) \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} Hd(\omega)\, e^{+j\omega n}\, d\omega$$

$$hd(n) \triangleq \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\omega\alpha}\, e^{j\omega n} . d\omega \rightarrow (1)$$

$$hd(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{+j\omega(n-\alpha)}\, d\omega$$

$$hd(n) = \frac{1}{2\pi} \left. \frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)} \right]_{-\omega_c}^{\omega_c}$$

$$hd(n) = \frac{1}{\pi(n-\alpha)} \left[\frac{e^{j\omega_c(n-\alpha)} - e^{-j\omega_c(n-\alpha)}}{2j} \right]$$

$$hd(n) = \frac{1}{\pi(n-\alpha)} \sin \omega_c(n-\alpha)$$

$$hd(n) = \frac{\sin \omega_c(n-\alpha)}{\pi(n-\alpha)} ; n \neq \alpha$$

Letting $n=\alpha$ in eqn(1), we get

$$hd(\alpha) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1 \cdot d\omega \qquad e^0 = 1$$

$$hd(\alpha) = \frac{2\omega_c}{2\pi} = \frac{\omega_c}{\pi}$$

Summarising the results, we get

$$hd(n) = \begin{cases} \frac{\sin \omega_c(n-\alpha)}{\pi(n-\alpha)} & ; n \neq \alpha \\ \frac{\omega_c}{\pi} & ; n = \alpha \end{cases}$$

Impulse response of low pass FIR filter is obtained by multiplying hd(n) by a causal window functn w(n) that starts at $n=0$ and ends at $n=N-1$

i.e $h(n) = hd(n)\,w(n) \quad ; 0 \leq n \leq N-1$

Since hd(n) is symmetric about $n=\alpha$ and window functn w(n) is symmetric about $n=\frac{N-1}{2}$

a linear phase results if

$$\alpha = \frac{N-1}{2}.$$

The condtn $\alpha = \frac{N-1}{2}$ ensures h(n) is symmetric about its mid pt.

Following imp points may be noted:

→ The cut-off freqencies ω_c depends upon the type of the window used

→ Transition width of FIR filter is approximately equal to width of main lobe of freqncy response of window used

Foll. table gives approximately transitn width and minimum stop band attenuatn for diff types of windows used in FIR filter table.

Type	transitn width ($\Delta \omega$) in rad	min. attenuatn in stop band
① Rectanglr	$\frac{4\pi}{N}$	21 dB
② Bartlett	$\frac{8\pi}{N}$	25 dB
③ Hanning	$\frac{8\pi}{N}$	44 dB
④ Hamming	$\frac{8\pi}{N}$	53 dB
⑤ Blackman	$\frac{12\pi}{N}$	74 dB

→ Let K_p, ω_p and K_s, ω_s represent the pass and stop band specificatn respectively. Then an FIR fltr is designed using the following iterative procedure:

① Select the type of window to be the one highest of the list, such that the stop band attenuatn exceeds $-K_s$ dB

② Size of the window is found using the eqn:

$\omega_s - \omega_p \geq K\frac{2\pi}{N}$; where $K = 2$ for rectangular

$K = 4$ for bartlett, hanning and hamming window

and $K = 6$ for blackmann window

③ Impulse response of low pass FIR filter is given by

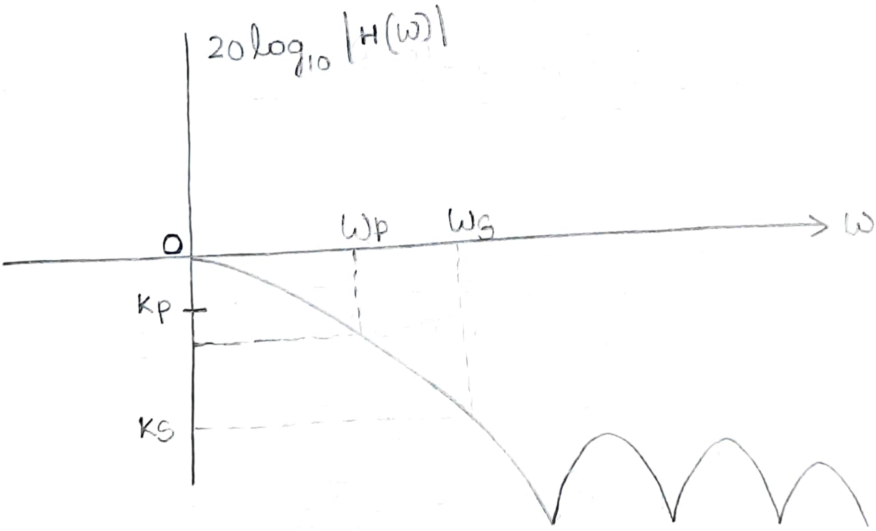
$$h(n) = hd(n)\,\omega(n) \quad ; 0 \le n \le N-1 \quad \rightarrow ①$$

In eqn ①, $hd(n) = \begin{cases} \dfrac{\sin W_c(n-\alpha)}{\pi(n-\alpha)} & ; n \ne \alpha \\ \dfrac{W_c}{\pi} & ; n = \alpha \end{cases}$

For 1st trial impulse response, we choose $\alpha = \frac{N-1}{2}$ and $W_c = W_p$. Since N is odd, freqncy response of FIR filter is computed using expresn given below:

$$H(w) = e^{-jw\left(\frac{N-1}{2}\right)} \left\{ h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{N-3/2} 2h(n) \cos\left[w\left(\frac{N-1}{2} - n\right)\right]\right. \quad \rightarrow ②$$

Using eqn ① and ②, we plot first trial freqncy response and it look approximately as shown in fig below:



As expected, at $w = W_p$, we get a pass band attenuatn greater than $-K_p$ dB. Hence, we increase W_c slightly and then using eqn ① and ②. Second trial freqncy response FIR filter is plotted. Again check for pass band requirement. If not satisfied. Repeat this procedure. Til pass-band and stop-band requirement are met

→ Once W_c is fixed, try decreasing N, so that passband and stop band requirement are not disturbed. In this way N and α are optimised.

→ Sub the values of W_c and optimised α in eqn ① and compute FIR filter co-efficient $h(n)$ for $n=0,1,\ldots N$.

① Frequency response of desired low pass filter is given by

$$H_d(\omega)=\begin{cases} e^{-j3\omega} & ; \ 0<|\omega|<\frac{\pi}{4} \\ 0 & ; \ \frac{\pi}{4}\ |\omega|<\pi \end{cases}$$

Design an FIR filter using hamming window. Also find an expresn for freqncy response of designed FIR filter?

S: WKT $hd(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi} Hd(\omega)\, e^{j\omega n}.\, d\omega$

$$\Rightarrow hd(n)=\frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{j\omega(n-3)}\, d\omega$$

$$hd(n)=\frac{\cancel{1}}{\cancel{2}\pi}\ \frac{\sin\frac{\pi}{4}(n-3)}{\pi(n-3)} \quad ; \ n\neq 3$$

letting $n=3$ in eqn ① we get

$$hd(n)=\frac{W_c}{\pi}=\frac{\pi}{4\times\pi}=\frac{1}{4}$$

Only for low pass filter

Summarising the results we get

$$hd(n)=\begin{cases}\frac{\sin\frac{\pi}{4}(n-3)}{\pi(n-3)} & ; \ n\neq 3 \\ 1/4 & ; \ n=3\end{cases}$$

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From exp for $Hd(\omega)$, we find that $\alpha = 3$

$\therefore \alpha = \frac{N-1}{2} = 3 \Rightarrow \underline{\underline{N = 7}}$

Impulse response of low pass FIR filter is

$$h(n) = hd(n)\, W_{ham}(n) \quad ; 0 \le n \le 6$$

$$W_{ham}(n) = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right) & ; 0 \le n \le N-1 \\ 0 & ; \text{otherwise} \end{cases}$$

n	hd(n)	Wham(n)	h(n)
0	0.075	0.08	0.006
1	0.15915	0.31	0.04933
2	0.22508	0.77	0.17325
3	1/4	1	1/4 = 0.25
4	0.22508	0.77	0.17325
5	0.15915	0.31	0.04933
6	0.075	0.08	0.006

Since N is odd, frequency response of low pass FIR filter is calculated using the expression given below:

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{N-3/2} 2h(n) \cos\left[\omega\left[\left(\frac{N-1}{2}\right) - n\right]\right]\right]$$

$N = 7$

$$\Rightarrow H(\omega) = e^{-j3\omega}\left[h(3) + \sum_{n=0}^{2} 2h(n) \cos[\omega(3 - n)]\right]$$

$$H(\omega) = e^{-j3\omega}\left[h(3) + 2h(0)\cos 3\omega + 2h(1)\cos 2\omega + 2h(2)\cos\omega\right]$$

$$H(\omega) = e^{-j3\omega}\left[0.25 + 0.012\cos 3\omega + 0.09866\cos 2\omega + 0.3466\cos\omega\right]$$

2) An FIR filter is specified by the following unit sample response:

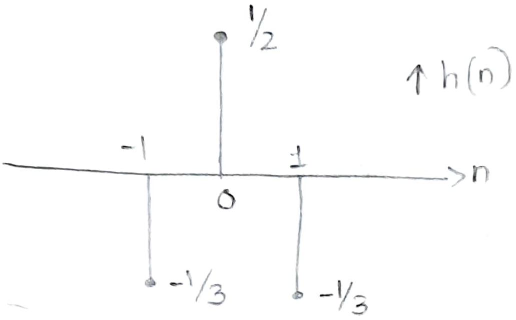
$$h(n) = -\frac{1}{3}\delta(n+1) + \frac{1}{2}\delta(n) - \frac{1}{3}\delta(n-1)$$

(a) Is it a ~~symmetry filter~~ and linear phase fltr? Expln

(b) Is it a causal filter? why or why not?

(c) Is it a low pass filter? If not, find the type of filter

S:



(a) Since $h(n)$ exhibits even symmetry about $n=0$ [mid pt], FIR filter will give a linear phase

(b) Since $h(n) \neq 0$ for $n<0$, filter is non causal.

(c) DTFT of $h(n)$ is

$$H(e^{jw}) = H(w) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-jwn}$$

$$\Rightarrow H(w) = \sum_{n=-\infty}^{\infty} \left[-\frac{1}{3}\delta(n+1) + \frac{1}{2}\delta(n) - \frac{1}{3}\delta(n-1)\right] e^{-jwn}$$

By applying sifting rule, we get

$$\Rightarrow H$$

$$H(w) = -\frac{1}{3} e^{-jwn}\Big|_{n=-1} + \frac{1}{2} e^{-jwn}\Big|_{n=0} - \frac{1}{3} e^{-jwn}\Big|_{n=1}$$

$$H(w) = -\frac{1}{3} e^{+jw} + \frac{1}{2} - \frac{1}{3} e^{-jw}$$

$$H(w) = \frac{1}{2} - \frac{1}{3}[2\cos w]$$

$$\Rightarrow |H(w)| = \left|\frac{1}{2} - \frac{2}{3}\cos w\right|$$

=

13, 27, 32, 43, 58

Rec, Bart, hamn, hamm, blac

$H(jw)$ gain

21, 25, 44, 53, 74

$W_R(n) = \{1, 0 \le n \le N-1$

$W_b(n) = \{1 - 2\frac{[n - (\frac{N-1}{2})]}{N-1}$

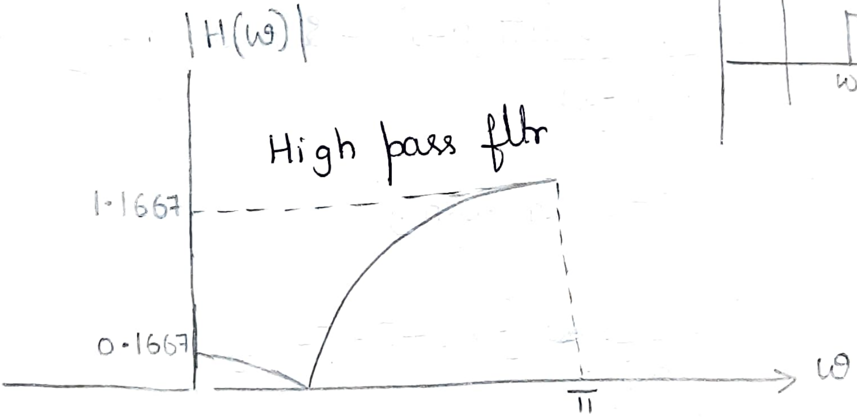
$W_{hn}(n) = 0.5 - 0.5\cos\left(\frac{2\pi n}{N}\right)$

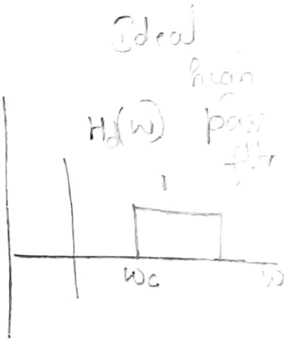
$W_{hm}(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)$

$W_{blac}(n) = 0.42 + 0.5 \ldots + 0.8\, C$

ω	$\lvert H(\omega)\rvert$
0	+0·16667
0·1π	+0·13403
0·2π	+0·03934
0·3π	0·10814
0·4π	0·2939
0·5π	0·5
0·6π	0·7060
0·7π	0·8918
0·8π	1·0393
0·9π	1·134
π	1·16667

→ 0·23π = 0





3) Design a high pass FIR fltr using Hammng window. Jake N=7 and wc = 2 rad. Also find an expression for freqncy response of designed high pass FIR fltr?

S: Frequency response of an ideal high pass filter is

$$H_d(w) = \begin{cases} e^{-jw\alpha} & ; \ w_c < |w| < \pi \\ 0 & ; \ 0 < |w| < w_c \end{cases}$$

To find hd(n):

$$hd(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} Hd(\omega) e^{+j\omega n} \cdot d\omega$$

$$hd(n) = \frac{1}{2\pi} \int_{-\pi}^{-\omega c} e^{+j\omega(n-\alpha)} \cdot d\omega + \int_{\omega c}^{\pi} e^{j\omega(n-\alpha)} \cdot d\omega \rightarrow (1)$$

$$hd(n) = \frac{1}{2\pi} \left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)} \right]_{-\pi}^{-\omega c} + \frac{1}{2\pi} \left(\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)} \right)_{\omega c}^{\pi}$$

$$= \frac{1}{2\pi} \left[\frac{e^{-j\omega c(n-\alpha)} - e^{-j\pi(n-\alpha)} + e^{j\pi(n-\alpha)} - e^{j\omega c(n-\alpha)}}{j(n-\alpha)} \right.$$

$$= \frac{1}{\pi(n-\alpha)} \left[\left[\frac{e^{j\pi(n-\alpha)} - e^{-j\pi(n-\alpha)}}{2j} \right] - \left(\frac{e^{j\omega c(n-\alpha)} - e^{-j\omega c(n-\alpha)}}{2j} \right) \right.$$

$$= \frac{1}{\pi(n-\alpha)} \left[\sin\pi(n-\alpha) - \sin\omega c(n-\alpha) \right] \quad ; n \neq \alpha$$

letting $n=\alpha$ in eqn (1)

$$hd(n) = \frac{1}{2\pi} \times \left[-\omega c + \pi + \pi - \omega c \right]$$

$$hd(n) = \left[\frac{\pi - \omega c}{\pi} \right] //$$

Summarising the results, we get

$$hd(n) = \begin{cases} \dfrac{\sin\pi(n-\alpha) - \sin\omega c(n-\alpha)}{\pi(n-\alpha)} & ; n \neq \alpha \\ \dfrac{\pi - \omega c}{\pi} & ; n = \alpha \end{cases}$$

Impulse response of high pass FIR fltr is

$$h(n) = hd(n)\, w(n)$$

where
$$W_{ham}(n) = \begin{cases} 0.54 - 0.46 \cos\left(\dfrac{2\pi n}{N-1}\right) & 0 \leq n \leq N-1 \\ 0 \; ; \; O.W \end{cases}$$

n	hd(n)	Wham(n)	h(n)
0	0.02964	0.08	0.00237
1	0.12045	0.31	0.0373
2	-0.2894	0.77	-0.2228
3	0.3633	1	0.3633
4	-0.2894	0.77	-0.2228
5	0.12045	0.31	0.0373
6	0.2894	0.08	0.00237

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FIR fltr Co-eff

and h(n) is even symmetric about its midpt

Since N is odd, frequency response of the high pass filter → FIR is

$$H(w) = e^{-jw\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{N-3/2} 2h(n)\cos w\left[\left(\frac{N-1}{2}\right)-n\right]\right]$$

$$\Rightarrow H(w) = e^{-3wj}\left[h(3) + \sum_{n=0}^{2} 2h(n)\cos[w(3-n)]\right]$$

$$\Rightarrow H(w) = e^{-j3w}\left[h(3) + 2h(0)\cos 3w + 2h(1)\cos 2w + 2h(2)\cos w\right]$$

$$\Rightarrow H(w) = e^{-j3w}\left[0.3633 + 0.00474\cos 3w + 0.0746\cos 2w - 0.4456\cos w\right]$$

4) Design a band-pass FIR fltr for the following specificatn: N = 7, w_{c_1} = 1 rad, w_{c_2} = 2 rad also find the magnitude of frequency response at w = 1.5 rad in dB. Use hanning window?

S:

Frequency response of ideal band pass filter is

$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha} & ; \omega_{c1} < |\omega| < \omega_{c2} < \pi \\ 0 & ; \text{otherwise} \end{cases}$$

WKT

$$h_d(n) \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} \cdot d\omega$$

$$h_d(n) = \frac{1}{2\pi} \int_{-\omega_{c2}}^{-\omega_{c1}} e^{-j\omega(n-\alpha)} \, d\omega + \int_{\omega_{c1}}^{\omega_{c2}} e^{-j\omega(n-\alpha)} \rightarrow (1)$$

$$h_d(n) = \frac{1}{\pi(n-\alpha)} \left[\sin \omega_{c2}(n-\alpha) - \sin \omega_{c1}(n-\alpha) \right] \; ; \; n \neq \alpha$$

Letting $n = \alpha$ in eqn (1) we get

$$h_d(n) = \frac{1}{2\pi} \left[-\omega_{c1} + \omega_{c2} + \omega_{c2} - \omega_{c1} \right]$$

$$h_d(n) = \frac{\omega_{c2} - \omega_{c1}}{\pi}$$

Impulse response of the band pass FIR fltr is given by

$$h(n) = h_d(n) W_{han}(n) \; ; \; 0 \leq n \leq N-1 = 6$$

Here

$$W_{han}(n) = \begin{cases} \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{N-1}\right)\right] & ; 0 \leq n \leq N-1 \leq 6 \\ 0 & ; \text{O.W} \end{cases}$$

$$\alpha = \frac{N-1}{2} = \frac{6}{2} = 3$$

n	hd(n)	Whan(n)	h(n)
0	-0.0446	0	0
1	-0.2651 ~~0.0258~~	0.25	-0.066275
2	0.02159 ~~1.17714~~	0.75	0.0161925
3	0.3183	1	0.3183
4	0.02159	0.75	0.0161925
5	-0.2651	0.25	-0.066275
6	-0.0446	0	0

Since N is odd and h(n) exhibits even symmetry frequency response of the band pass FIR filter is

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$$H(w) = e^{-jw\left(\frac{N-1}{2}\right)} \left\{ h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} 2h(n) \cos w\left[\left(\frac{N-1}{2}\right) - n\right] \right.$$

$$|H(w)| = \left| h\left(\frac{N-1}{2}\right) \right| + \left| \sum_{n=0}^{\frac{N-3}{2}} 2h(n) \cos\left(w(3-n)\right) \right|$$

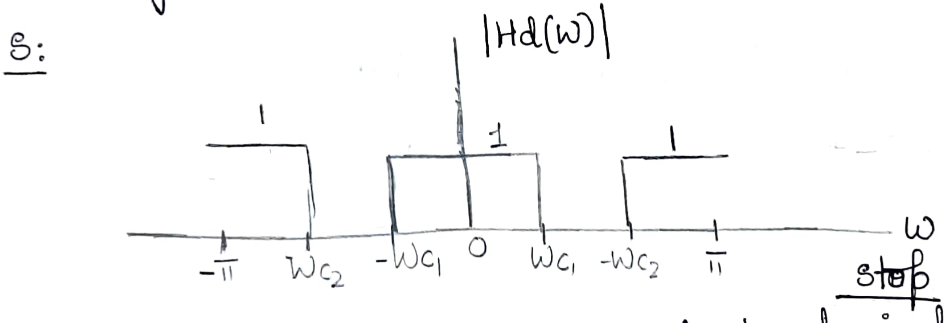
$$|H(w)| = \left| h(3) + 2h(0)\cos 3w + 2h(1)\cos 2w + 2h(2)\cos w \right|$$

$$|H(w)| = \left| 0.3183 - 0.1326 \cos 2w + 0.03238 \cos w \right|$$

$$20 \log_{10} |H(w)| \Big|_{w=1.5} = -6.609 \text{ dB}$$

5) Design a band reject FIR filter to meet the following specifications: $w_{c_1} = 1$ rad, $w_{c_2} = 2$ rad $N = 7$. Use rectangular window?

S:



Frequency response of ideal band reject filter is

$$Hd(w) = \begin{cases} e^{-jw\alpha} & ; \text{otherwise} \\ 0 & ; w_{c_1} < |w| < w_{c_2} \end{cases}$$

WKT

$$hd(n) \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} Hd(w) e^{jwn} dw$$

$$hd(n) \triangleq \frac{1}{2\pi} \int_{-\pi}^{-w_{c_2}} e^{jw(n-\alpha)} dw + \frac{1}{2\pi} \int_{-w_{c_1}}^{w_{c_1}} e^{jw(n-\alpha)} dw +$$

$$\frac{1}{2\pi} \int_{w_{c_2}}^{\pi} e^{+jw(n-\alpha)} dw$$

$$hd(n) = \frac{1}{2\pi} \left(\left[\frac{e^{jw(n-\alpha)}}{j(n-\alpha)} \right]_{-\pi}^{-w_{c_2}} + \left[\frac{e^{jw(n-\alpha)}}{j(n-\alpha)} \right]_{-w_{c_1}}^{w_{c_1}} + \left[\frac{e^{jw(n-\alpha)}}{j(n-\alpha)} \right] \right.$$

$$\Rightarrow hd(n) = \frac{1}{\pi(n-\alpha)}\left[\sin\pi(n-\alpha) + \sin w_{c1}(n-\alpha) - \sin w_{c2}(n-\alpha)\right]$$
$$; n \neq \alpha$$

letting $n=\alpha$ we get

$$hd(n) = \frac{1}{2\pi}\left[\int_{-\pi}^{-w_{c2}} dw + \int_{-w_{c1}}^{w_{c1}} dw + \int_{w_{c2}}^{\pi} dw\right]$$

$$= \frac{-w_{c2}+\pi+w_{c1}+w_{c1}+\pi-w_2}{2\pi}$$

$$hd(n) = \frac{w_{c1}-w_2+\pi}{\pi} \quad n=\alpha$$

Hence, the impulse response of band reject FIR filter is

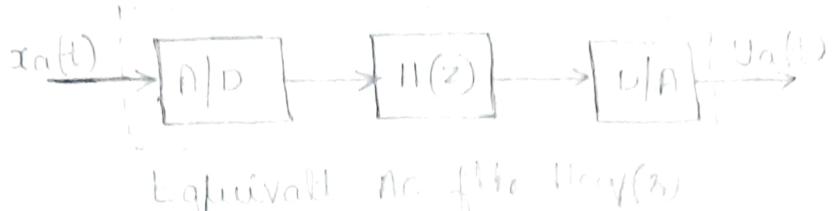
$$h(n) = hd(n)\, w_n(n) \quad ; 0 \leq n \leq N-1=6$$

where $w_n(n) = \begin{cases} 1, & 0 \leq n \leq 6 \\ 0, & o.w \end{cases}$

n	hd(n)	wn(n)	h(n)
0	+0.044	1	+0.044
1	0.265	1	0.265
2	−0.0215	1	−0.0215
3	0.6817	1	0.6817
4	0.0215	1	−0.0215
5	0.265	1	0.265
6	+0.044	1	0.044

(67) Design a low pass digital filter to be used in A/D – H(z) – D/A structure that will have – 3dB cut-off at 30π rad/s and an attenuatn of 50 dB at 45π rad/s. Filter is required to have a linear phase and system uses sampling rate of 100 sample per second.

S:



In the problem, specificatn Heq(s) are given and they are as follows:

$K_p = -3\,dB$, $\Omega_p = 30\pi\ rad/s$

$K_s = -50\,dB$, $\Omega_s = 45\pi\ rad/s$ and $T = \frac{1}{100}$ sec

① Convert band edge analog freqncy into dig freqncies using the formula

$$W = \Omega T \;;\; T = 1/100\ s$$

$$W_p = \Omega_p T = 0.3\pi \;;\; K_p = -3\,dB$$

$$W_s = \Omega_s T = 0.45\pi \;;\; K_s = -30\,dB$$

② ~~Prewarp~~

Type	transitn width (ΔW)	min stop-band atte
Rectanglar	$4\pi/N$	21dB
Bartlet	$8\pi/N$	25 dB
Hanning	$8\pi/N$	44 dB
Hamming	$8\pi/N$	53 dB
Blackman	$12\pi/N$	74 dB

To meet a stop band attenuation of 50dB, we have two choices in the form of hamming and blackman window. Since blackman has higher transition width compare hamming window, we choose hamming window

③ Size of the window is selected using the relation given below

$$W_s - W_p \geq \frac{2\pi}{N} \cdot K \qquad 4 \times \frac{2\pi}{N} = \frac{8\pi}{N}$$

हेतु hamming window ; $K = 4$

hence $0.45\pi - 0.3\pi \geq \frac{8\pi}{N}$

$$N \geq \frac{8}{0.15}$$

$$N \geq 53.33$$

Selecting N to be next higher odd integer

$N = 55$ $\left[\alpha = \frac{N-1}{2} = 27\right\}$ must be an integer$]$

④ let the frequency response of low pass ideal filter be

$$Hd(\omega) = \begin{cases} e^{-j\omega\alpha} & ; |\omega| < \omega_c \\ 0 & ; \omega_c < |\omega| < \pi \end{cases}$$

$$hd(n) \triangleq \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} Hd(\omega) e^{j\omega n} d\omega \qquad e^{-j\omega\alpha}$$

$$hd(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega(n-\alpha)} d\omega$$

$$hd(n) = \frac{\sin \omega_c(n-\alpha)}{\pi(n-\alpha)} \; ; \; n \neq \alpha$$

letting $n = \alpha$ in eqn ①, we get

$$hd(\alpha) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1 \cdot d\omega = \frac{\omega_c}{\pi}$$

Impulse response of low pass FIR filter is

$h(n) = hd(n)\, W_{ham}(n) \;;\; 0 \le n \le N-1 = 54$

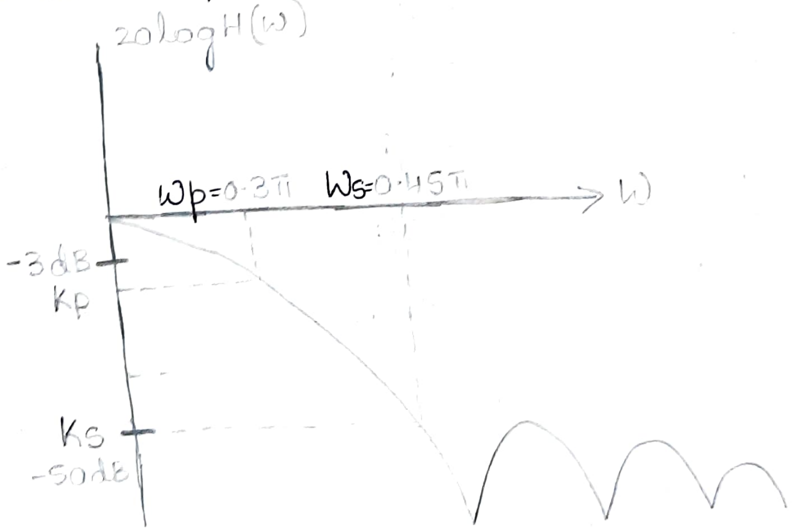
$$\Rightarrow h(n) = \begin{cases} \dfrac{\sin W_c(n-\alpha)}{\pi(n-\alpha)} \times 0.54 - 0.46\cos\left(\dfrac{2\pi n}{N-1}\right) & ; n \neq \alpha \\ \dfrac{W_c}{\pi} \times 1 & ; n = \alpha \end{cases} \quad \to (2)$$

For first trial, impulse response, we choose $W_c = W_p = 0.3\pi$ and $\alpha = \frac{N-1}{2} = 27$

Since N is odd, freqncy response of low pass FIR filter is computed using the expresn given blw

$$H(w) = e^{-jw\left(\frac{N-1}{2}\right)} \left\{ h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{N-3/2} 2h(n) \cos w\left(\left(\frac{N-1}{2}\right) - n\right) \right. \quad \to (3)$$

Using eqn (2) and (3), first trial freqncy response is plotted and it appears as shown blw



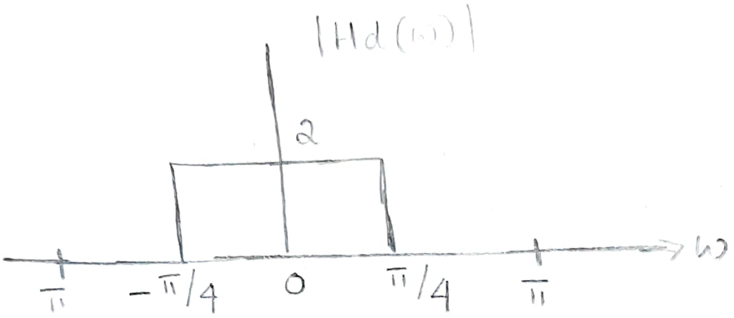
It is found from above fig that $W_p = 0.3\pi$, the pass band attenuatn is greater than -3dB. Hence, slightly increase W_c and then eqn (2) and (3) plot mag response. check for pass band requirement. Continue this procedure till at $W_p = W_p$. the pass band attenuatn less than or equal -3dB (matlab ans $W_c = 0.33\pi$)

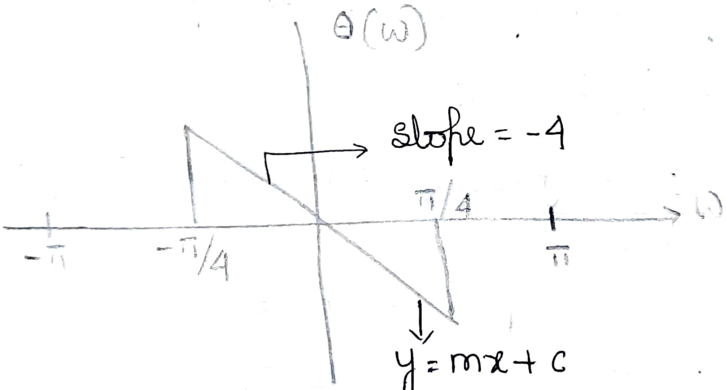
7) Fig below shows magnitude and phase respon of an ideal low pass filter. Find hd (n)

b) Suppose hd (n) is trunkated by a hamming window. What is h(n)?

c) Roughly sketch the freqncy response of low pass FIR fltr designed in part b. Ensure to indicate the values of cut-off freqncy and transitn and stop band attenuatn

Sl:





S: In general, Hd(ω) is complex and is given by

$$Hd(\omega) = |Hd(\omega)| e^{j\theta(\omega)}$$

$$= \begin{cases} 2e^{-j4\omega} & ; -\frac{\pi}{4} \leq |\omega| < \frac{\pi}{4} \\ 0 & ; \frac{\pi}{4} < |\omega| < \pi \end{cases}$$

WKT

$$hd(n) \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} Hd(\omega) e^{j\omega n} d\omega$$

$$hd(n) = \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} 2e^{j\omega(n-4)} \cdot d\omega \rightarrow (1)$$

$$hd(n) = \frac{2 \sin \frac{\pi}{4}(n-4)}{\pi(n-4)} ; n \neq 4$$

letting $n=4$ in eqn ① we get

$$h_d(n) = \frac{1}{2\pi}\left[\frac{2\pi}{4}\right] = \frac{1}{2} \times 1 = \frac{1}{2}$$

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(b) WKT

$h(n)$

Impulse response of low pass FIR filter is

$$h(n) = h_d(n)\, w_{hamm}(n) \quad ; 0 \le n \le N-1 = 8$$

[NOTE : $\alpha = 4 \Rightarrow \frac{N-1}{2} = \alpha \Rightarrow N = 9$]

where, $w_{Ham}(n) = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right) & ; 0 \le n \le N-1 = 8 \\ 0 & ; \text{otherwise} \end{cases}$

n	$h_d(n)$	$w_{ham}(n)$	$h(n)$
0	0	0.08	0
1	0.15	0.2147	0.0322
2	0.318	0.54	0.17172
3	0.45	0.8653	0.3893
4	1/2	1	0.5
5	0.45	0.8653	0.3893
6	0.318	0.54	0.17172
7	0.15	0.2147	0.0322
8	0	0.08	0

(c) Since N is odd and $h(n)$ has even sym

IIR Filter design.

characteristics of commonly used analog filters - Butterworth & Chebyshev filters. Analog to analog freq transformations.

Introduction:

A Filter is one which rejects unwanted freq's from the i/p signals & allows the desired freq's to obtain the required (shape) o/p signal

Types of Filters

Filter < analog filter, digital filter

analog filter - i/p & o/p are continuous time signal

Digital filter - " " " discrete-time signals

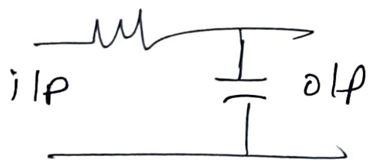
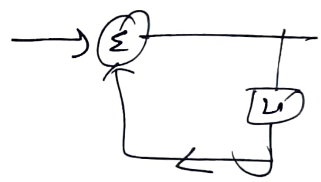
* A digital filter is generally a discrete LTI system which approximates a freq response ie, desired with its i/p being digital samples & o/p also digital samples

* types of Digital Filter

IIR (Infinite impulse Response)	FIR [Finite Impulse Response]
are recursive type where present o/p depends on present i/p, past i/p & o/p sample	are non recursive type where present o/p depends upon present i/p & past i/p samples

Classification of Analog Filter in accordance with the freq selective characteristics as
Low pass filter. Band pass
High pass filter & band stop filter

Comparision bet' Analog & Digital filter.

Analog	Digital
① I/O's are continuous time signals	① I/O's are discrete time signals
② Implementation of these filters is carried out using passive components	② These are implemented on a digital computer or H/W using DSP elements such as adder, x^r & delay
	
③ Analog filter theory is based on linear differential eqn	③ based on linear difference eqn
④ Laplace transforms are used for analysis in 's'-plane	④ Z-transforms are used for analysis in Z-plane
⑤ disadv/ higher noise sensitivity, non linearity, lack of flexibility	⑤ Digital filter requires additional ADC/DAC converters. freq range is restricted to half the sampling rate
⑥ environmental, parameters, interference, & noise affects the performance	⑥ Negligible effect of environmental parameters & interference noise & others

* In this chapter the design of IIR filter that are realizeable & stable are discussed in detail.

* The impulse response h(n) for a realizable filter is

$$h(n) = 0 \; ; \; n \leq 0 \rightarrow (1)$$

& for stability it must satisfy the condition

$$\sum_{n=0}^{\infty} |h(n)| < \infty \rightarrow (2)$$

IIR digital filters have transfer fun of the form

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \frac{\sum_{K=0}^{M} b_K z^{-K}}{1 + \sum_{K=1}^{N} a_K z^{-K}} \rightarrow (3)$$

* The design of an IIR filter for given specifications is finding filter coefficients a_Ks & b_Ks of eq (3).

RQ types of Analog filter:

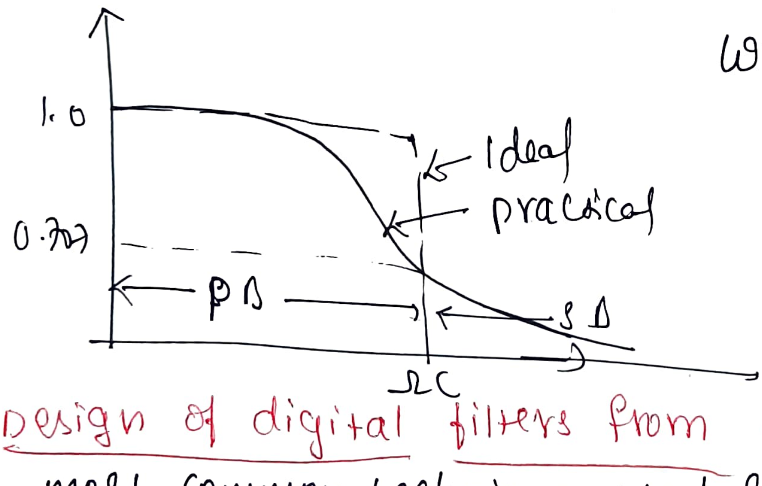
A filter is one which rejects unwanted freq from the i/p signal & allows the desired freq's to obtain the required shape of o/p signal

Pass band → The range of freq's of signal that passed thro' the filter

stop band → those freq's that are blocked

* The Analog filters are of different types based on
  1. their magnitude response - LP, HP, BP, BS
  2. their cut off freq
  3. shape of their amplitude response
  4. shape of their phase response
  5. nature of device used < Passive / Active

① Low pass filter - allows Low freq's to pass thro' it while attenuates high freq



Design of digital filters from Analog filter

The most common technique used for designing IIR digital filters is known as indirect method - which involves first designing an analog prototype filter & then transforming the prototype to a digital filter

* for given specifications of digital filter, the derivation of digital filter Transfer fun requires 3 steps

(1) Map the desired digital filter specifications into equivalent Analog filter

(2) derive the analog TF for the analog prototype

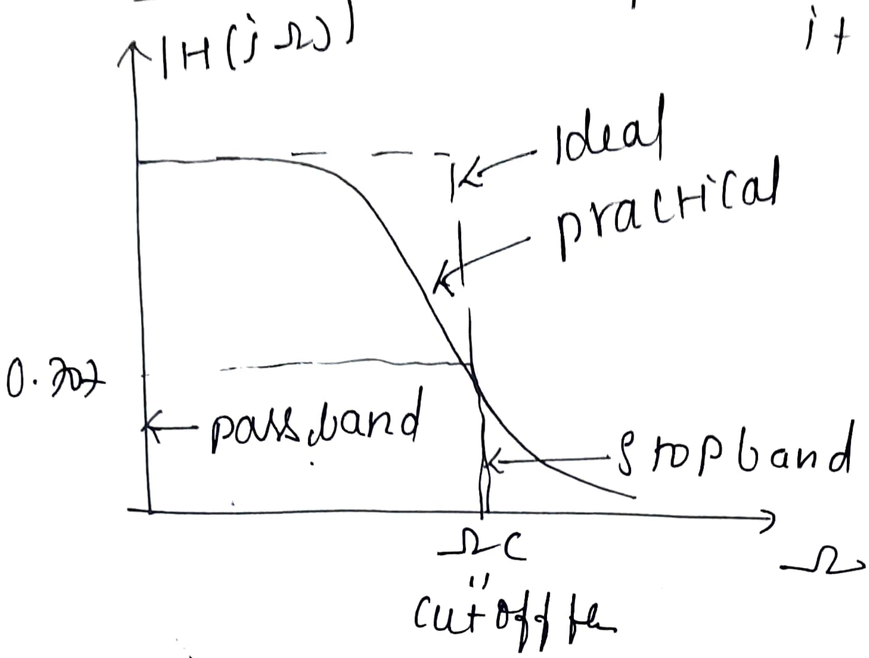
(3) Transform the TF of Analog prototype into an equivalent digital filter transfer fun

Ideal & practical response of LP, HP, BP & BS filter

(3)

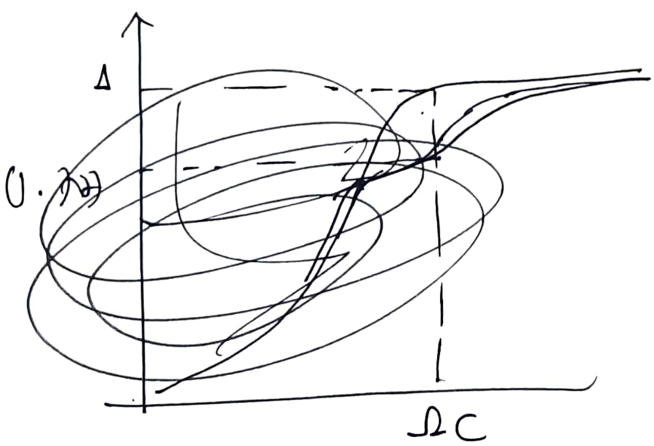
(1) Low pass :-

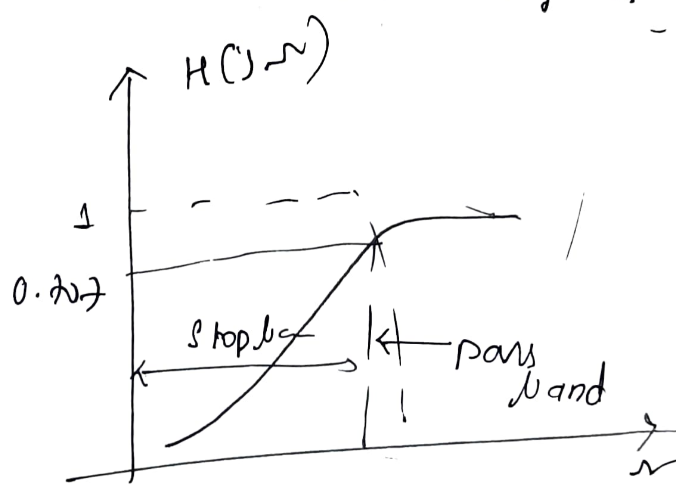
allows Low freq's to pass thro' it while it attenuates high freq.



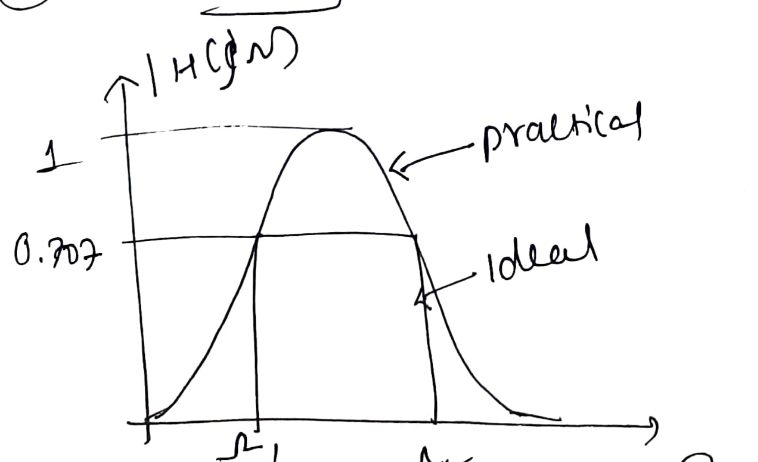
Cut off freq: defined as freq where the gain has changed by some specified amount relative to mean mid band gain

(2) High pass



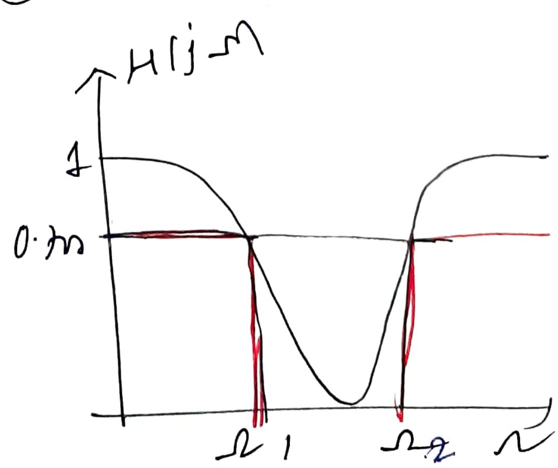


(3) Band pass



It allows only a band of freqs Ω1 to Ω2 to pass & stops all other freq

(4) Band-Reject

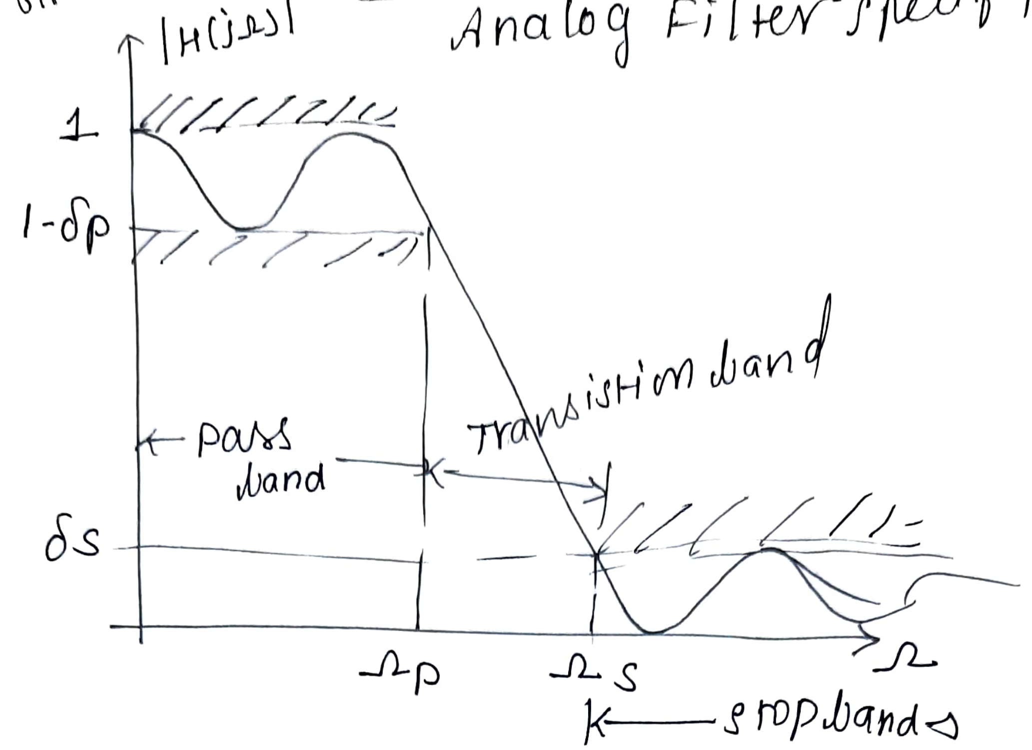


It rejects all freq's bet' Ω1 & Ω2 & allows remaining fre

GR

IIR-filter

Analog Filter specify



Specification of LPF

* An important step in the design of an analog filter is the definition of the freq response specifications that should be ~~sto~~ satisfied by the filter freq response

* These specifications describe how the steady-state filter reacts in the to sinusoidal i/p

* fig above shows a typical magnitude freq response to LPF

---

pass band → The range of freq's of signal that are passed thro' the filter

stop band → those freq's that are blocked

$\Omega_p \rightarrow$ pass band edge freq (4)

$\Omega_s \rightarrow$ stop band edge freq.

* The freq range bet' Ω_p & Ω_s is called transition band while no specification is provided.

* The hatched area indicates forbidden magnitude values in these bands.

* In transition band, the magnitude decreases monotonically in this band

* mathematical description of the freq response is

$$\boxed{\begin{aligned} 1-\delta_p \leq |H(j\Omega)| \leq 1, &\quad \text{for } 0 \leq \Omega \leq \Omega_p \\ 0 \leq |H(j\Omega)| \leq \delta_s &\quad \text{for } |\Omega| \geq \Omega_s \end{aligned}}$$

* $\delta_p \rightarrow$ tolerance of magnitude response in pass band.

* The desired magnitude response in passband is 1.

~~$\delta_p \Rightarrow$ pass band toleran~~'

* $\delta_s \rightarrow$ tolerance of magnitude response in stopband.

* The desired magnitude response in the stopband is 0.

* we define $\delta p \rightarrow$ ~~pass band atten.~~

① $$\boxed{A_p = -20 \log (1-\delta p)}$$

as passband ripple in dB.

② $$\boxed{K_p = -A_p = 20 \log (1-\delta p)}$$

is defined as passband gain at $\Omega = \Omega_p$.

$\delta s \rightarrow$ stopband attenuation.

③ $\boxed{A_s = -20 \log \delta s.}$ is defined as stopband attenuation or ripple in dB

④ $$\boxed{K_s = -A_s = 20 \log_{10} \delta s} \Big|_{\Omega = \Omega_s}$$

is defined as stopband gain at $\Omega = \Omega_s$.

Note:- $\delta p \rightarrow$ pass band tolerance / ripple
$\delta s \rightarrow$ stopband " "
$A_p \rightarrow$ pass band attenuation
$A_s \rightarrow$ stop band "

* The main classes of analog filter

Butterworth filter Chebyshev filter

* In this section the properties & design procedures for analog filters (Butterworth & Chebyshev) are presented along with the procedures & transform required to convert them into LPF, HPF, BPF, BSF.

Butterworth Filters

[All pole approximation / butterworth app / monotonic pass band & monotonic stop band]

They have smooth pass band with a relatively wide transition regn. where as chebyshev filter have sharp transition region & a not so smooth pass band.

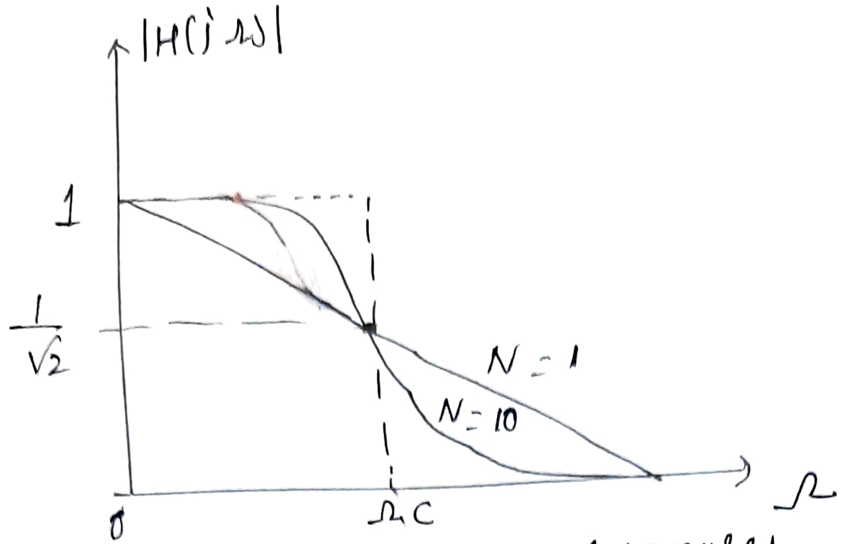
* A Butterworth filter is characterized by its magnitude freq response

$$|H(j\Omega)| = \frac{1}{\left[1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right]^{1/2}} \rightarrow (1)$$

where

$N \rightarrow$ order of the filter

$\Omega_c \rightarrow$ cutoff freq where the filter magnitude is $\frac{1}{\sqrt{2}}$ times the dc gain ($\Omega=0$)



Typical magnitude responses of butterworth filter

The foll observations are made from the above fig:.

1) $|H(j0)| = 1$ for all N $\Omega = 0$

2) $|H(j\Omega)| = \frac{1}{\sqrt{2}}$ at $\Omega = \Omega_c$ for all finite N.

This means that $20\log_{10} |H(j\Omega_c)| = -3.01dB$

3) $|H(j\Omega)| \to 0$ as $\Omega \to \infty$

4) The magnitude characteristics is said to be maximally flat $\therefore$

$$\left.\frac{d^n |H(j\Omega)|}{d\Omega^n}\right|_{\Omega=0} = 0 \quad \text{for } n = 1, 2, \dots 2N-1.$$

5) $|H(j\Omega)|$ is a monotonically decreasing fun of freq (Ω)

i.e, $|H(j\Omega_2)| < |H(j\Omega_1)|$ for any values of Ω_1 & Ω_2 such that $0 \le \Omega_1 < \Omega_2$

6

* The magnitude-square freq response of the normalized ($\Omega_c = 1$) LP Butterworth filter is

$$\boxed{|H_N(j\Omega)|^2 = \frac{1}{1+\Omega^{2N}}} \quad \text{— (1)}$$

$|A| = A \times A^*$

$$H_N(j\Omega)\, H_N(-j\Omega)| = \frac{1}{1+\Omega^{2N}} \rightarrow (2)$$

replacing $j\Omega$ by s & hence $\Omega = \frac{s}{j}$ in eq (2) we get

$$H_N(s)\, H_N(-s) = \frac{1}{1+\left(\frac{s}{j}\right)^{2N}} \rightarrow (3)$$

* The transfer fun $H_N(s) \cdot H_N(-s)$ has no finite zeros. ~~∴ $H_N(s)$ itself has no fi~~

* The poles of the product $H_N(s)\, H_N(-s)$ are determined by equating the denominator to zero

i.e, $1 + \left(\frac{s}{j}\right)^{2N} = 0$

$$\left(\frac{s}{j}\right)^{2N} = -1$$

$$\left(\frac{s}{j}\right) = (-1)^{1/2N}$$

$\cancel{S = (-1)^{1/N}}$

$$\boxed{S = (-1)^{1/2N} \cdot j} \longrightarrow (H)$$

WKT $\boxed{\begin{aligned} -1 &= e^{j\pi(2K+1)} \\ \& \ j &= e^{j\pi/2}. \end{aligned}}$ $K = 0, 1, \sim 2N-1$

The poles are given by

$$S_K = e^{\frac{j\pi(2K+1)}{2N}} \cdot e^{j\pi/2}$$

$$S_K = e^{j\left(\frac{K\pi}{N} + \frac{\pi}{2N} + \frac{\pi}{2}\right)} \longrightarrow (5)$$

$$= 1 e^{j\theta_K}$$

$$= 1 \angle \theta_K$$

The poles of $H_N(s) H_N(-s)$ are the roots of the characteristic eqn that lie on the circle of unit radius & are placed ~~with~~ at. angles

$$\boxed{\theta_K = \frac{K\pi}{N} + \frac{\pi}{2N} + \frac{\pi}{2}} \quad 0 \le K \le (2N-1)$$

$\longrightarrow (6)$

* The above analysis implies that the roots of the characteristic eqn are the poles of $H_N(s)\, H_N(-s)$ which lies on a unit circle.

* The transfer fun of normalized LP butterworth filter is

$$\boxed{H_N(s) = \frac{1}{\underset{\substack{\text{LHP} \\ \downarrow}}{\prod} (s - s_k)} = \frac{1}{B_N(s)}}$$

s_k = left hand poles

$B_N(s)$ = Butterworth polynomial of order N.

Table below shows first 5 Butterworth polynomial in a real factured form

N	$B_N(s)$
1	$s+1$
2	$s^2 + \sqrt{2}s + 1$
3	$(s^2+s+1)(s+1)$
4	$(s^2+0.76536s+1)(s^2+1.84776s+1)$
5	$(s+1)(s^2+0.6180s+1)(s^2+1.6180s+1)$

1) N = 1 [First order filter]

WKT

$S_K = 1 \angle \theta_K = e^{j\theta_K}$

$$\boxed{\theta_K = \frac{K\pi}{N} + \frac{\pi}{2N} + \frac{\pi}{2} \quad K = 0, 1, \dots 2N-1}$$

for $N = 1$, $K = 0, 1$

$\therefore \theta_K = K\pi + \frac{\pi}{2} + \frac{\pi}{2}$

$$\boxed{\theta_K = K\pi + \pi}$$

(i) K = 0 $S_0 = 1 \angle \theta_0 =$

$S_0 = 1 \angle \theta_0$

$\theta_0 = (0)\pi + \pi = \pi$

$S_0 = 1 \angle \theta_0 = 1 \angle \pi = 1e^{j\pi} = \cos\pi + j\sin\pi$

$-1 + j0$

$$\boxed{S_0 = -1}$$

(ii) K = 1

$S_1 = 1 \angle \theta_1$

$\theta_1 = \pi + \pi = 2\pi$

$S_1 = 1 \angle 2\pi = 1e^{j2\pi} = \cos 2\pi + j\sin 2\pi$

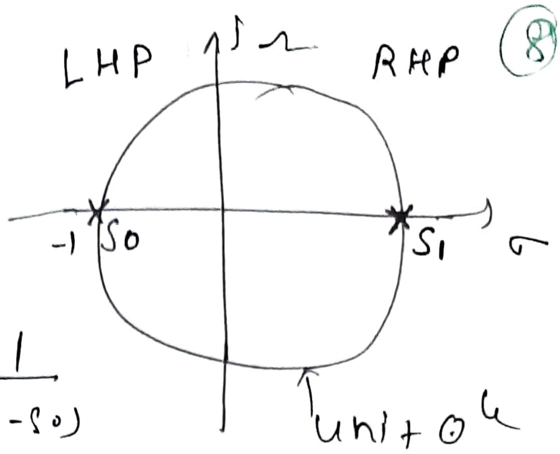
$1 + j0$

$\boxed{S_1 = 1}$

(8)

$B_N(s) = \prod (s - s_k)$

$= \prod (s - s_0)$



$B_1(s) = s - (-1)$ $\qquad H_1(s) = \frac{1}{(s - s_0)} = \frac{1}{s+1}$

$\boxed{B_1(s) = (s+1)}$

Cross check

$H_1(j\Omega) = \frac{1}{j\Omega + 1}$

$|H_1(j\Omega)| = \frac{1}{\sqrt{\Omega^2 + 1}}$

$\Omega_c = 1$

$|H_1(j\Omega)|_{\Omega=1} = \frac{1}{\sqrt{2}}$

2) N = 2 (2nd order filter)

for $N = 2$ $\qquad K = 0, 1, 2, 3$

$\therefore \theta_K = \frac{K\pi}{N} + \frac{\pi}{2N} + \frac{\pi}{2}$

$= \frac{K\pi}{2} + \frac{\pi}{4} + \frac{\pi}{2}$

$\boxed{\theta_K = K\frac{\pi}{2} + \frac{3\pi}{4}}$ $\qquad K = 0, 1, 2, 3$

(i) $\cancel{s_k}\,K = 0$ $\qquad s_0 = 1\angle\theta_0$

$\theta_0 = 0 \cdot \frac{\pi}{2} + \frac{3\pi}{4} = \frac{3\pi}{4}$

$s_0 = 1\angle 3\pi/4 = \cos 135 + j\sin 135$

$\boxed{s_0 = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}}$

(ii) $K = 1$ $\qquad s_1 = 1\angle\theta_1 = 1\angle\left(\frac{\pi}{2} + \frac{3\pi}{4}\right) = 1\angle\frac{5\pi}{4}$ $\qquad \boxed{\frac{5\pi}{4} = 225^\circ}$

$= \cos\frac{5\pi}{4} + j\sin\frac{5\pi}{4}$

$\boxed{s_1 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}}$

(iii) $S_2 = 1\angle\theta_2$ $\theta_2 = \frac{2\pi}{2} + \frac{3\pi}{4} = \frac{7\pi}{4} = 315°$

$$S_2 = 1\angle 7\pi/4 = \cos\frac{7\pi}{4} + j\sin\frac{7\pi}{4}$$

$$\boxed{s_2 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}}$$

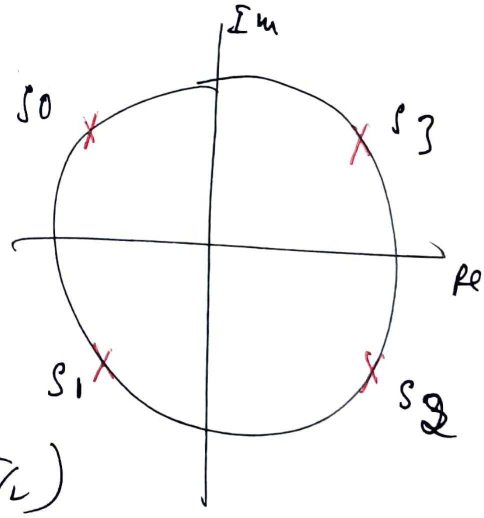
(iv) $\underline{K=3}$ $s_3 = 1\angle\theta_3$

$$\theta_3 = \frac{3\pi}{2} + \frac{3\pi}{4} = \frac{9\pi}{4} = 405°$$

$$S_3 = 1\angle\frac{9\pi}{4} = \cos\frac{9\pi}{4} + j\sin\frac{9\pi}{4}$$

$$\boxed{s_3 = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}}$$

$$H_2(s) = \frac{1}{(s-s_0)(s-s_1)}$$



$$B_2(s) = (s-s_0)(s-s_1)$$

$$= \left(s+\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)\left(s+\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)$$

$$= s^2 + \frac{s}{\sqrt{2}} + \cancel{j\frac{s}{\sqrt{2}}} + \frac{s}{\sqrt{2}} + \frac{1}{2} + \cancel{j\frac{1}{2}} - \cancel{j\frac{s}{\sqrt{2}}} - \cancel{j\frac{1}{2}} + \frac{1}{2}$$

$$= s^2 + \frac{2s}{\sqrt{2}} + 1$$

$$\boxed{B_2(s) = s^2 + \sqrt{2}s + 1}$$

$$H_2(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

(9)

(iii) $\underline{N = 3}$ (Third order filter)

WKT $\quad S_K = 1 \angle \theta_K$

$$\theta_K = \frac{K\pi}{N} + \frac{\pi}{2N} + \frac{\pi}{2}; \quad K = 0, 1, \ldots 2N-1$$

$N = 3$ we get

$$\theta_K = \frac{K\pi}{3} + \frac{\pi}{6} + \frac{\pi}{2}; \quad K = 0, 1, 2, 3, 4, 5$$

$$= \frac{K\pi}{3} + \frac{2\pi}{3}$$

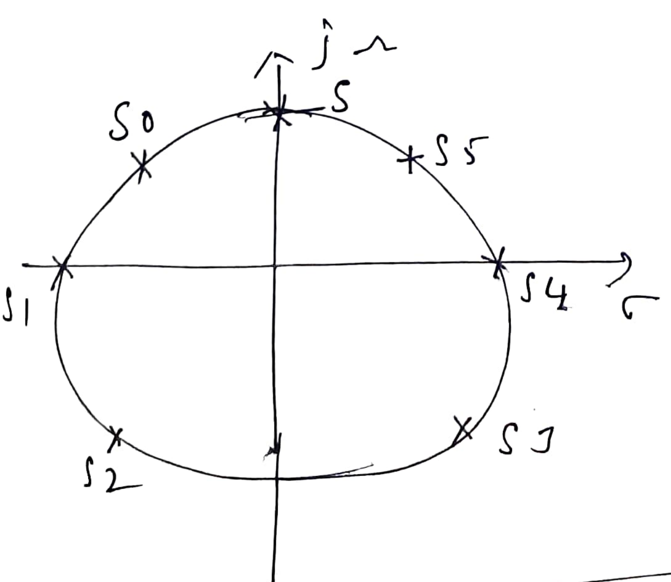
$$S_0 = 1 \angle 2\pi/3 = -0.5 + j\,0.866$$
$(\sqrt{3}/2)$

$$S_1 = 1 \angle \pi = -1$$

$$S_2 = 1 \angle 4\pi/3 = -0.5 - j\,0.866$$

$$S_3 = 1 \angle 4\pi/3 = -0.5 - j\,0.866$$

$$S_4 = 1 \angle 2\pi = 1, \quad S_5 = 1 \angle 7\pi/3 = 0.5 + j\,0.866$$



$$H_3(s) = \frac{1}{(s - s_0)(s - s_1)(s - s_2)}$$

$$B_3(s) = (s + 0.5 - j\,0.866)(s+1)(s + 0.5 + j\,0.866)$$

$$= (s+1)\left[(s+0.5)^2 + (0.866)^2\right]$$

$$\boxed{B_3(s) = (s+1)(s^2 + s + 1)}$$

Frequency Transformation or Spectral transformation or Analog to Analog Transformation

* A LPF is used as a prototype or standard for all types of filter design

* Let $H(s)$ be the transfer fun of normalised LPF

* Let $H'(s)$ is a transfer fun of a new filter which is obtained by replacing

$$s \text{ by } \frac{s}{\Omega_u}$$

i.e, $H'(s) = H(s)\Big|_{s \to \left(\frac{s}{\Omega_u}\right)}$

$$= H\left(\frac{s}{\Omega_u}\right)$$

Let $s = j\Omega$

$$H'(j\Omega) = H\left(\frac{j\Omega}{\Omega_u}\right)$$

Let $\Omega = \Omega_u$

$$H'(j\Omega_u) = H(j)$$

* The above eqn means that the freq response (10)
of a new filter evaluated at $\Omega = \Omega_u$.
is equal to the value of the normalized
filter at $\Omega = 1$ rad/sec

* The cut off freq has moved from
$\Omega_c = 1$ rad/sec to Ω_u rad/sec

* lly transformation can be defined
for normalized LPF to
HPF, BPF & BSF as shown below

1) normalized Low pass to Low pass transform

$$\boxed{s \to \frac{s}{\Omega_u}}$$

2) Low pass to high pass transformation

$$\boxed{s \to \frac{\Omega_u}{s}}$$

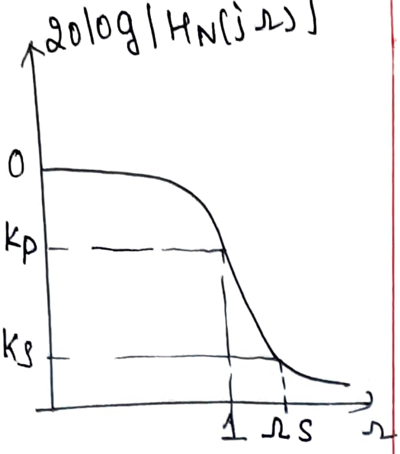
3) Lowpass to band pass transformation
[combination of LP & HP]

$$\boxed{s \to \frac{s^2 + \Omega_u \Omega_l}{s(\Omega_u - \Omega_l)}}$$

4) Lowpass to band stop transformation

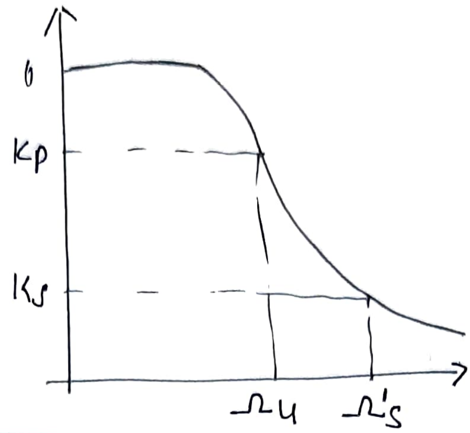
$$\boxed{s \to \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l}}$$

Prototype freq response	Transformed freq response	Backward design eqn



LPF $H_N(s)$

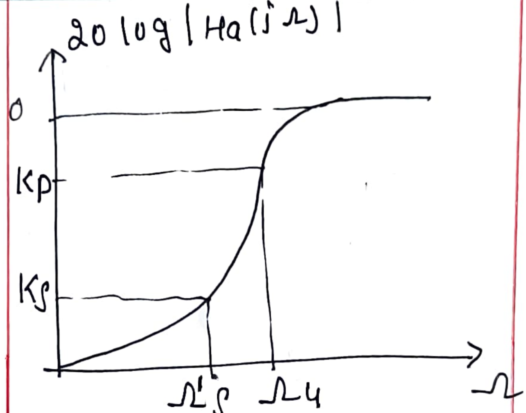
$\boxed{s \to \frac{s}{\Omega_u}}$



LPF $H_a(s)$

$$\Omega_s = \frac{\Omega'_s}{\Omega_u}$$

— " —



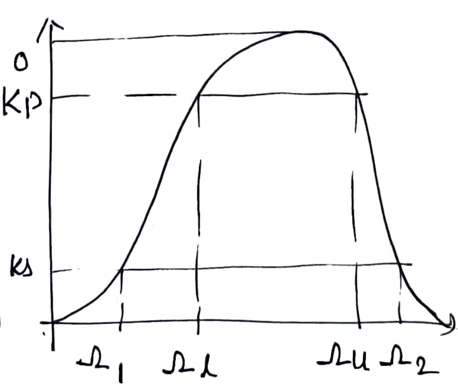
$\boxed{s \to \frac{\Omega_u}{s}}$

HPF

$$\Omega_s = \frac{\Omega_u}{\Omega'_s}$$

— " —

$$s \to \frac{s^2+\Omega_u\Omega_l}{s(\Omega_u-\Omega_l)}$$



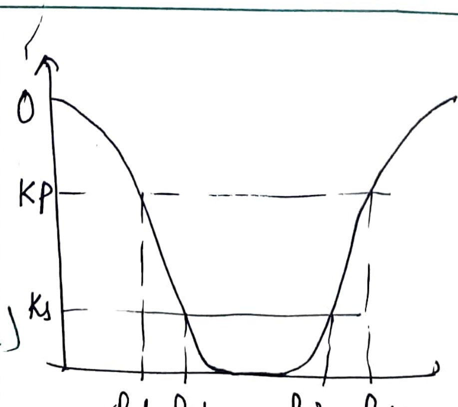
$$\Omega_s = \min\{|A|, |B|\}$$

$$A = \frac{-\Omega_1^2+\Omega_l\Omega_u}{\Omega_1(\Omega_u-\Omega_l)}$$

$$B = \frac{\Omega_2^2-\Omega_l\Omega_u}{\Omega_2(\Omega_u-\Omega_l)}$$

— " —

$$s \to \frac{s(\Omega_u-\Omega_l)}{s^2+\Omega_u\Omega_l}$$

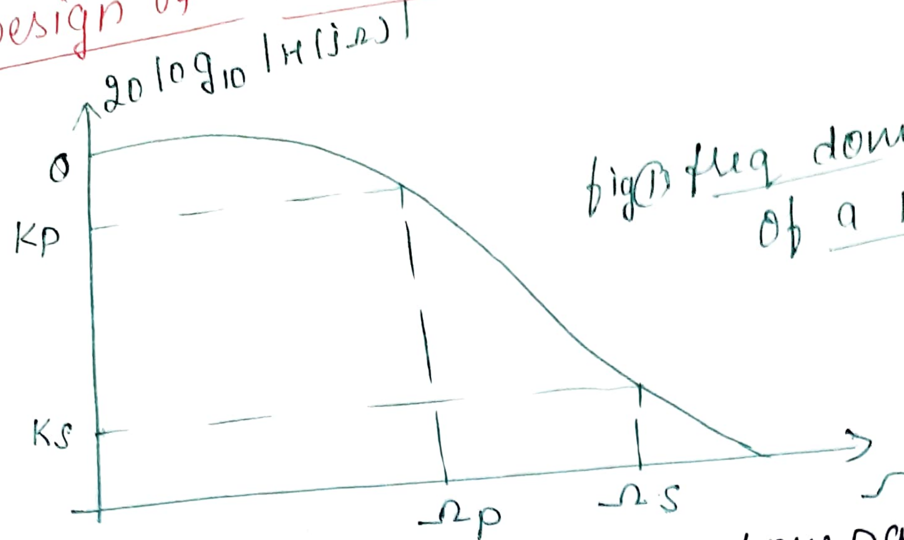


$$\Omega_s = \min\{|A|, |B|\}$$

$$A = \frac{\Omega_1(\Omega_u-\Omega_l)}{-\Omega_1^2+\Omega_l\Omega_u}$$

$$B = \frac{\Omega_2(\Omega_u-\Omega_l)}{\Omega_2^2-\Omega_u\Omega_l}$$

Design of Butterworth



fig(1) freq domain specification of a L.P Butterworth filter

- we are required to design a Low pass Butterworth filter given the foll specifications.

pass band gain $- K_p$, stop band gain $- K_s$
pass band freq $- \Omega_p$ & stop band freq $- \Omega_s$

$$K_p \le 20 \log |H(j\Omega)| \le 0 \quad \text{for all } \Omega \le \Omega_p$$

$$\& \quad 20 \log |H(j\Omega)| \le K_s \quad \text{for all } \Omega \ge \Omega_s$$

The magnitude response of a LP Butterworth filter is given by

$$|H(j\Omega)| = \frac{1}{\left[1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}\right]^{1/2}} \longrightarrow (1)$$

-

to find N:-

taking $20\log_{10}$ on both sides we get

$$20\log_{10} |H(j\Omega)| = -20\log_{10}\left[1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}\right]^{1/2}$$

$$= -10\log_{10}\left[1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}\right] \longrightarrow (2)$$

from fig ① we find that

at $\Omega = \Omega_p$, $20 \log_{10} |H(j\Omega)| = K_p$

$\therefore$ eq ② becomes

$$K_p = -10 \log_{10} \left[1 + \left(\frac{\Omega_p}{\Omega_c}\right)^{2N} \right]$$

$$\therefore \boxed{\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = 10^{-\frac{K_p}{10}} - 1} \rightarrow ③$$

IIIy at $\Omega = \Omega_s$, $20 \log_{10} |H(j\Omega)| = K_s$

eq ②

$$K_s = -10 \log_{10} \left[1 + \left(\frac{\Omega_s}{\Omega_c}\right)^{2N} \right]$$

$$\boxed{\left(\frac{\Omega_s}{\Omega_c}\right)^{2N} = 10^{-\frac{K_s}{10}} - 1} \rightarrow ④$$

divinding eq ③ by ④

$$\left(\frac{\Omega_p}{\Omega_s}\right)^{2N} = \frac{10^{-K_p/10} - 1}{10^{-K_s/10} - 1}$$

Taking logalithm on both side

$$2N \log_{10}\left(\frac{\Omega_p}{\Omega_s}\right) = \log_{10}\left[\frac{(10^{-K_p/10} - 1)}{(10^{-K_s/10} - 1)}\right]$$

$$\therefore \quad \boxed{N = \frac{\log_{10}\left[(10^{-K_P/10}-1)/(10^{-K_S/10}-1)\right]}{2\log_{10}\left(\frac{\Omega_P}{\Omega_S}\right)}} \rightarrow (5)$$

If N is an integer, we take that value otherwise N is rounded of to next larger integer

Once the order of 'N' is decided, the procedure for finding the cut off freq Ω_C is as follows:

1) If we desire to meet the pass band requirement exactly the cut off freq is selected from eq (3)

$$\left(\frac{\Omega_P}{\Omega_C}\right)^{2N} = 10^{-K_P/10} - 1$$

$$\therefore \quad \Omega_{C_1} = \frac{\Omega_P}{\left[10^{-\frac{K_P}{10}} - 1\right]^{1/2N}} \longrightarrow (6)$$

2) If we wish to meet our requirements at stop band then

$$\Omega_{C_2} = \frac{\Omega_S}{\left[10^{-K_S/10} - 1\right]^{1/2N}} \longrightarrow (7)$$

3) If one desires to do better in both the bands select Ω_C - cut off freq as the arithemetic mean of the 2 cut off freq's found above

$$\Omega_C = \underset{\text{mean}}{(\text{avg})}\ (\Omega_{C_1} \ \& \ \Omega_{C_2})$$

Design steps

1) from the given specifications, find the order of the filter 'N'

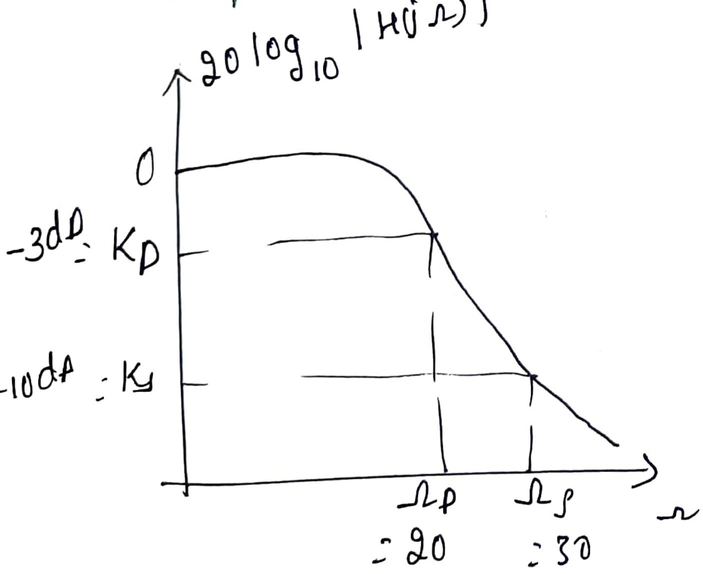
2) Find the cut off freq Ω_c to meet the PB/SB/ both requirements

3) to determine Butterworth polynomials & Transfer function for a normalized LP filter.

4) to find Transfer fun of the filter given

$$H_a(s) = H_N(s)\Big|_{s \rightarrow} \left(\begin{array}{c}\text{depanding} \\ \text{on filter}\end{array}\right)$$

① Design an Analog butterworth filter that has a -3dB pass band attenuation at a freq of 20 rad/sec and atleast -10dB stop band attenuation at 30 rad/sec



Given:

$K_p = -3dB \quad \Omega_p = 20 rad/sec$

$K_s = -10dB \quad \Omega_s = 30 rad/sec$

$$\left(\frac{\Omega_p}{\Omega_s}\right) = \frac{2}{3}$$

1) WKT

$$N = \frac{\log_{10}\left[(10^{-K_p/10} - 1) \Big/ (10^{-K_s/10} - 1)\right]}{2\log_{10}\left(\frac{\Omega_p}{\Omega_s}\right)}$$

$$= \frac{\log_{10}\left[\left(10^{-\left(\frac{-3}{10}\right)}-1\right) \Big/ \left(10^{-\left(\frac{-10}{10}\right)}-1\right)\right]}{2\log_{10}\left(\frac{20}{30}\right)}$$

$N = 2.7153$ Rounding off to the next larger integer

$\boxed{N = 3}$

(ii) Find normalised butterworth polynomial & Transfer fun

derive it

$$B_3(s) = (s+1)(s^2+s+1)$$

TF: $H_3(s) = \dfrac{1}{(s+1)(s^2+s+1)} = \dfrac{1}{B_3(s)}$

(iii) Let us determine the cut off the freq Ωc to meet the P.B

$$\Omega c = \frac{\Omega p}{\left[10^{-Kp/10}-1\right]^{1/2N}} = \frac{20}{\left[10^{-\left(\frac{-3}{10}\right)}-1\right]^{1/2\times 3}}$$

$\boxed{\Omega c = 20 \text{ rad/sec}}$

(iv) to find T.F for a cut off freq 20 rad/s

$$Ha(s) = H_3(s)\Big|_{s \to \frac{s}{\Omega c}} = H_3(s)\Big|_{s \to \frac{s}{20}}$$

$$H_a(s) = \frac{1}{\left[\frac{s}{20}+1\right]\left[\left(\frac{s}{20}\right)^2+\left(\frac{s}{20}\right)+1\right]}$$

$$= \frac{1}{\frac{s+20}{20}\left[\frac{s^2}{400}+\frac{20s}{400}+\frac{400}{400}\right]}$$

$$= \frac{20\times 400}{(s+20)(s^2+20s+400)}$$

$$\boxed{H_a(s) = \frac{8\times 10^3}{(s+20)(s^2+20s+400)}}$$

Verification of the design

Let $s = j\Omega$ in $H_a(s)$

$$H_a(j\Omega) = \frac{8\times 10^3}{(j\Omega+20)(-\Omega^2+j20\Omega+400)}$$

$$|H_a(j\Omega)| = \frac{8\times 10^3}{\sqrt{\Omega^2+400}\ \sqrt{(400-\Omega^2)^2+(20\Omega)^2}}$$

$$20\log_{10}|H_a(j\Omega)|_{\Omega=20}$$

$$= 20\log_{10}\left[\frac{\dfrac{8\times10^3}{\sqrt{20^2+400}}}{\sqrt{(400-400)^2+(20\times20)^2}}\right]$$

$$= -3.01\,dB$$

$$\simeq -3\,dB$$

$$20\log_{10}|H_a(j\Omega)|_{\Omega=30} = 20\log_{10}\left(\quad\right)$$

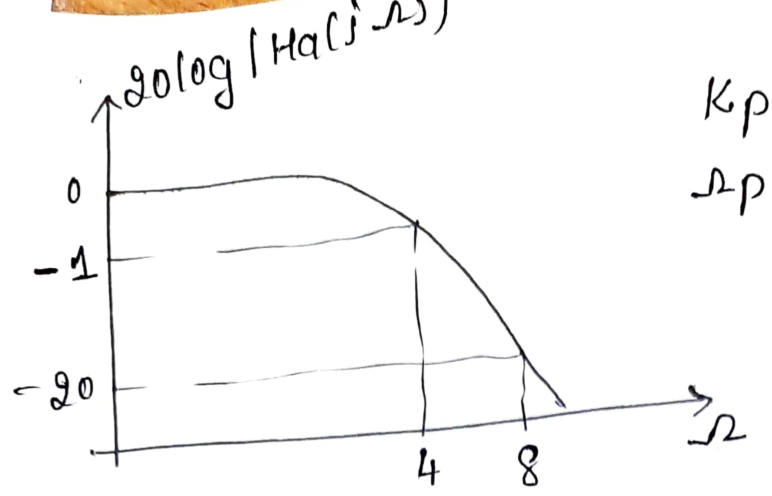
$$= -10.93\,dB$$

(2) A butterworth LPF has to meet the foll specifications

(a) pass band gain $K_p = -1\,dB$ at $\Omega_p = 4$ rad/sec

(b) stopband gain is greater than or equal to 20dB at $\Omega_s = 8$ rad/sec

determine the T.F $H_a(s)$ of the butterworth filter to meet the above specification

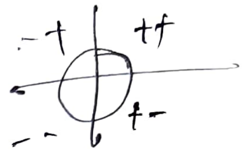


$K_p = -1dB$ $K_s \geq -20dB$

$\Omega_p = 4$ rad/sec $\Omega_s = 8$ rad/s

(i) $N = 4.289 \cong 5$

ii) to find 5th order normalised butterworth polynomial $B_5(s)$



w.k.t $S_K = 1\angle\theta_K$

$$\theta_K = \frac{\pi K}{N} + \frac{\pi}{2N} + \frac{\pi}{2}$$

$N = 5$

$S_0 = 1\angle\frac{6\pi}{10} = -0.309 + j0.951$

$S_1 = 1\angle 8\pi/10 = -0.809 + j0.588$

$S_2 = 1\angle\pi = -1$

$S_3 = 1\angle\frac{12\pi}{10} = -0.809 - j0.588$

$S_4 = 1\angle\frac{14\pi}{10} = -0.309 - j0.951$

$S_5 = 1\angle\frac{16\pi}{10} = 0.309 - j0.951$

$S_6 = 1\angle\frac{18\pi}{10} = 0.809 - j0.588$

$S_7 = 1\angle 2\pi = 1$

$S_8 = 1\angle\frac{22\pi}{10} = 0.809 + j0.588$

$S_9 = 1\angle 24\pi/10 = 0.309 + j0.951$

(15)

$$H_5(s) = \frac{1}{(s-s_0)(s-s_1)(s-s_2)(s-s_3)(s-s_4)} = \frac{1}{B_5(s)}$$

$$= \frac{1}{(s+1)\,(s^2+0.6180s+1)\,(s^2+1.6180s+1)}$$

$$= \frac{1}{s^5+3.236s^4+5.236s^3+5.236s^2+3.236s+1}$$

(iii) <u>to find cut off freq</u>

$$\Omega_{c_1} = \frac{\Omega_p}{\left(10^{-K_p/10}-1\right)^{1/2N}} = \cancel{4.5787\ rad/sec}\ 4.5787$$

$$\Omega_{c_2} = 5.05$$

$$\Omega_c = 4.815$$

(iv) <u>to find $H_a(s)$</u>

$$H_a(s) = H_5(s)\Big|_{s \to \frac{s}{\Omega_c}}$$

$$= \frac{1}{\left(\frac{s}{4.5787}\right)^5 + 3.236\left(\frac{s}{4.5787}\right)^4 + 5.236\left(\frac{s}{4.5787}\right)^3 + 5.236\left(\frac{s}{4.5787}\right)^2 + 3.236\,(s/4.5787) + 1}$$

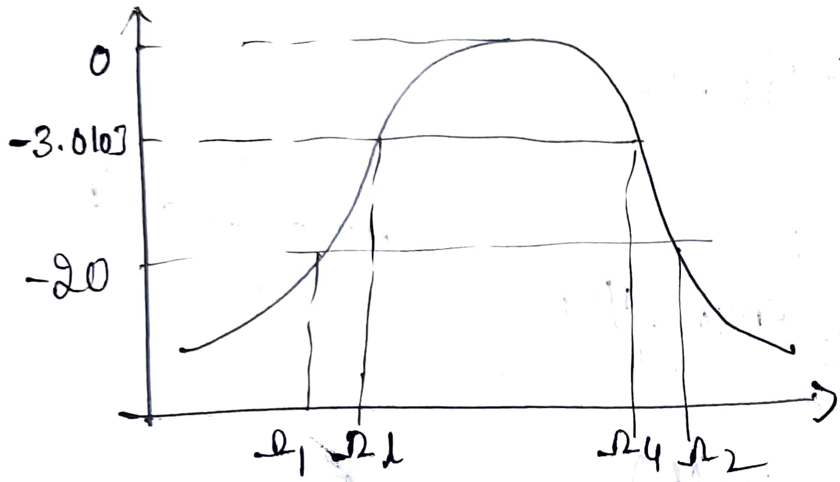
$$= \frac{2012.4}{s^5 + 14.82s^4 + 109.8s^3 + 502.6s^2 + 1422.3s + 2012.411}$$

(3) Design a analog band pass filter to meet the foll freq-domain specifications

(a) −3.0103 dB upper & lower cut off freq of 50 Hz & 20 KHz

(b) a stopband attenuation of atleast 20 dB at 20 Hz & 45 KHz

& (c) a monotonic freq response.



NOTE:- Ω values must be in radians

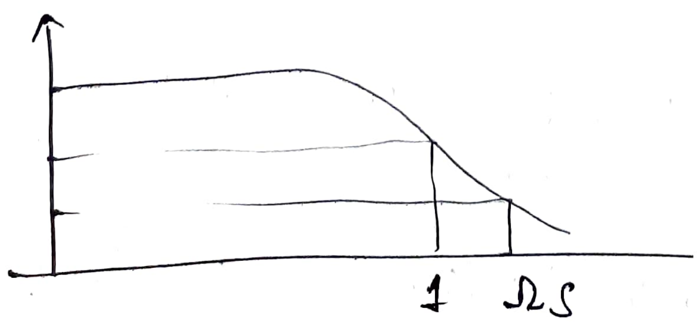
$\Omega_1 = 2\pi \times 20 = 125.663$ rad/sec

$\Omega_2 = 45 \times 10^3 \times 2\pi = 2.827 \times 10^5$ rad/sec

$\Omega_u = 2\pi \times 20 \times 10^3 = 1.257 \times 10^5$ rad/sec

$\Omega_l = 2\pi \times 50 = 314.159$ rad/sec

Let us use the backward design



$$A = \frac{-\Omega_1^2 + \Omega_l \Omega_u}{\Omega_1(\Omega_u - \Omega_l)} = 2.51$$

(16)

$$B = \frac{\Omega_2^2 - \Omega_l \Omega_u}{\Omega_2(\Omega_u - \Omega_l)} = 2.25$$

$$\Omega_s = \min[|A|, |B|] = \underline{\underline{2.25}}$$

ii) Order of normalised LP butterworth filter

$$N = \frac{\log\left[\left[10^{-\frac{K_p}{10}} - 1\right] / \left[10^{-\frac{K_s}{10}} - 1\right]\right]}{2\log\left(\frac{1}{\Omega_s}\right)}$$

$$= 2.83$$

$$N \approx \underline{\underline{3}}$$

(iii) $H_3(s) = \dfrac{1}{s^3 + 2s^2 + 2s + 1}$

(iv) $H_a(s) = H_3(s) \Big|_{s \to \frac{s^2 + \Omega_u \Omega_l}{s(\Omega_u - \Omega_l)}}$ $= \dfrac{s^2 + 3.949 \times 10^7}{(1.2538 \times 10^5)s}$

$\Omega_u \Omega_l = 3.949 \times 10^7$

$\Omega_u - \Omega_l = 1.2538 \times 10^5$

$$H_a(s) = \frac{1.9695 \times 10^{15}\, s^3}{s^6 + 2.51 \times 10^5 s^5 + 3.154 \times 10^5 s^4 + 1.989 \times 10^{15} s^3 + 1.2453 \times 10^{18} s^2 + 3.9073 \times 10^{20} s + 6.1529 \times 10^{22}}$$

(4) Let $H(s) = \frac{1}{s^2+s+1}$ represent the TF of a LPF with pass band freq of 1 rad/sec use freq transformations to find the T.Fs of the foll analog filter.

(a) A LPF with a pass band of 10 rad/sec

(b) A HPF with a cut off freq of 10 rad/sec

(c) A BPF with a pass band of 10 rad/sec & center freq of 100 rad/sec

(d) A band stop filter with a stop band of 2 rad/sec & a center freq of 10 rad/sec

$$H(s) = \frac{1}{s^2+s+1}$$

(a) $s \to \frac{s}{\Omega_u}$

$H_a(s) = H(s)\Big|_{s \to \frac{s}{10}}$

$$\boxed{H_a(s) = \frac{100}{s^2+10s+100}}$$

(b) $s \to \frac{\Omega_u}{s} = \frac{10}{s}$

$H_a(s) = H(s)\Big|_{s \to \frac{10}{s}}$

$$\boxed{H_a(s) = \frac{s^2}{s^2+10s+100}}$$

(c) ~~$s =$~~

(c) LPF → BPF

$$s \rightarrow \frac{s^2 + \Omega_u \Omega_l}{s(\Omega_u - \Omega_l)}$$

center freq of bandpass

$$\Omega_0 = \sqrt{\Omega_u \Omega_l}$$

B_0 = width of passband = $\Omega_u - \Omega_l$

$$s \rightarrow \frac{s^2 + \Omega_0^2}{sB_0} = \frac{s^2 + (100)^2}{10 \times s}$$

$$= \frac{s^2 + 10^4}{10s}$$

$$H_a(s) = \frac{100\, s^2}{s^4 + 10s^3 + 20100 s^2 + 10^5 s + 10^8}$$

(d) LPF → BSF

$$s \rightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l} = \frac{sB_0}{s^2 + \Omega_0^2}$$

$$s \rightarrow \frac{2s}{s^2 + 100}$$

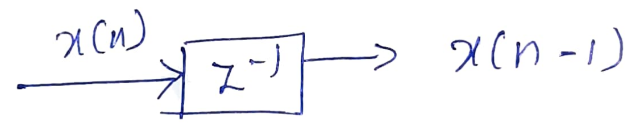
$$\boxed{H_a(s) = \frac{(s^2 + 100)^2}{s^4 + 2s^3 + 204 s^2 + 200 s + 10^4}}$$

* A Digital filters can be realized using various configurations or structures.

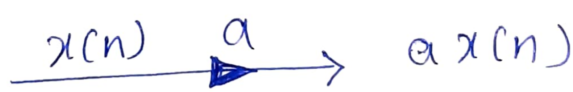
* A structure can be represented using either block diagram or signal flow graph

* The blockdiagram representation of a digital filter consists unit delays, multipliers, adders pick-off nodes are shown below

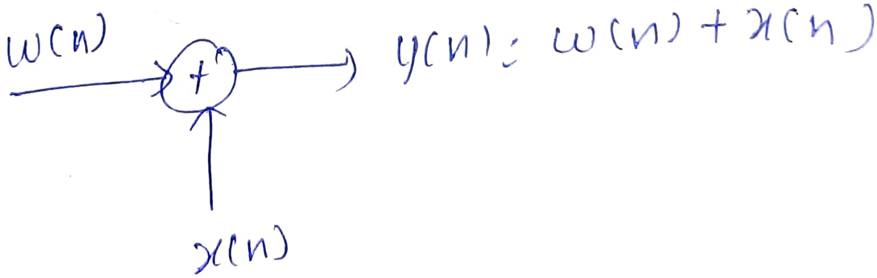
unit delay



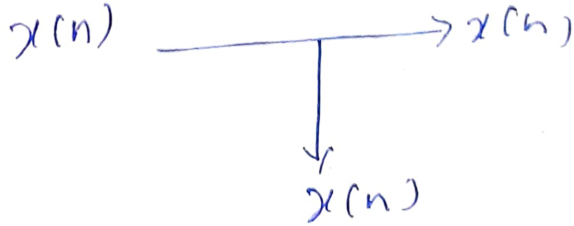
multipler



adder



pick-off node



Basic IIR Filter Structures

~~consider~~ causal IIR filter is characterized by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \rightarrow (1)$$

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \cdots + a_N z^{-N}} \rightarrow (2)$$

cross multiply & IZT

$$y(n) = -\sum_{k=1}^{N} a_k z^{-k} + \sum_{k=0}^{M} b_k z^{-k} \rightarrow (3)$$

$Y(z)$

1. Direct-form structure

we represent IIR filter in several form

(1) direct form I

(2) " " II

(3) cascade

(4) parallel

(5) lattice

(6) lattice lade

from eq (1)

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \rightarrow (1)$$

we can represent $H(z)$ as cascade of 2 systems ~~fn~~ with system fun $H_1(z)$ & $H_2(z)$

$$H(z) = H_1(z) \cdot H_2(z) \rightarrow (2)$$

$$X(z) \longrightarrow \boxed{H_1(z)} \xrightarrow{W(z)} \boxed{H_2(z)} \longrightarrow y(n)$$ (2)

where

$$H_1(z) = \frac{W(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k} \longrightarrow (3)$$

$$H_2(z) = \frac{Y(z)}{W(z)} = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}} \longrightarrow (4)$$

eg. consider a third order $(N=3)$ filter

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} \longrightarrow (5)$$

eq (5) can be written as

$$H(z) = H_1(z) \cdot H_2(z)$$

where

$$H_1(z) = \frac{W(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} \longrightarrow (6)$$

&

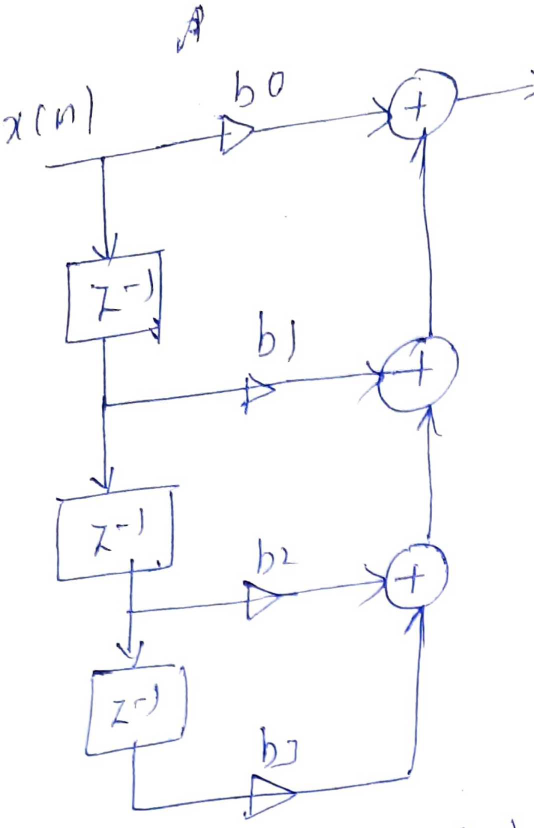
$$H_2(z) = \frac{Y(z)}{W(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} \longrightarrow (7)$$

Taking inverse ZT on eq (6)

$$\boxed{w(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + b_3 x(n-3)} \longrightarrow (8)$$

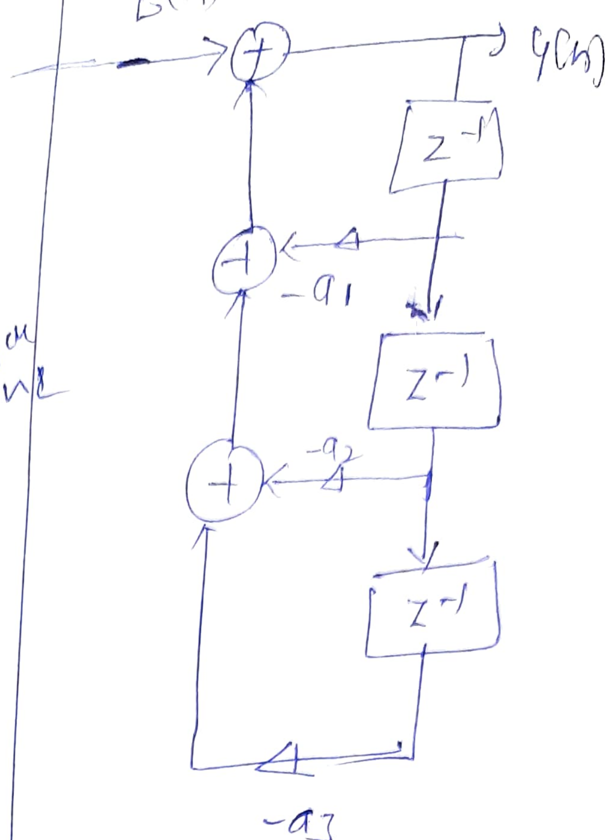
Taking inverse ZT on eq (7)

$$\boxed{y(n) = w(n) - a_1 y(n-1) - a_2 y(n-2) + a_3 y(n-3)} \longrightarrow (9)$$



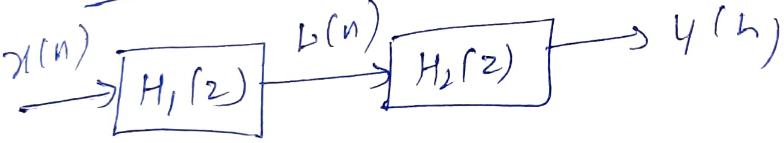
structure for $H_1(z)$

direct form II

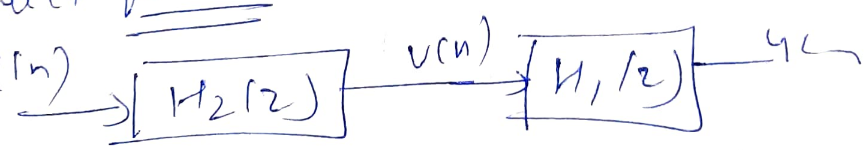


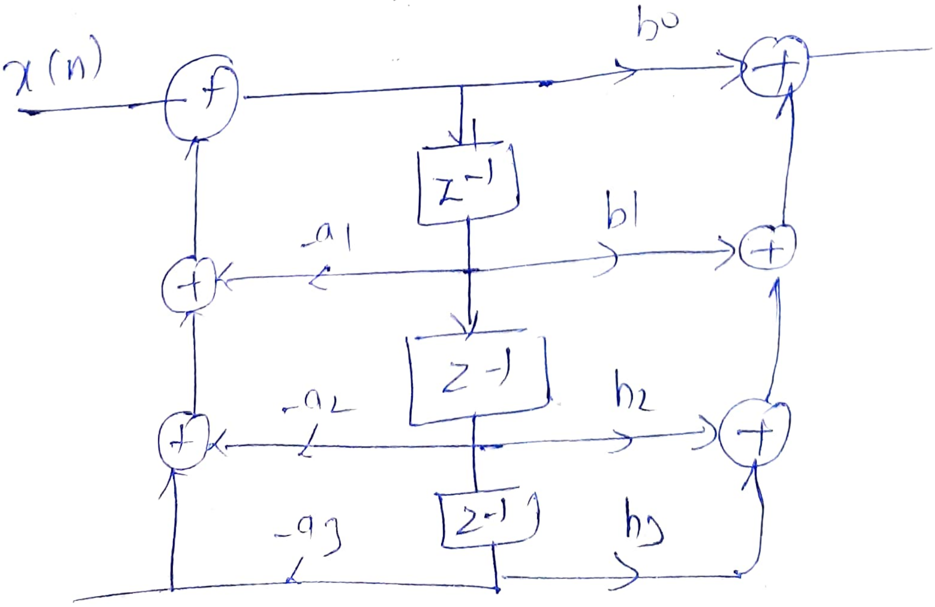
structure for $H_2(z)$

I direct form I



II direct form 2





I obtain direct form - I & II realizations (3)
for a digital IIR filter described by the system fun

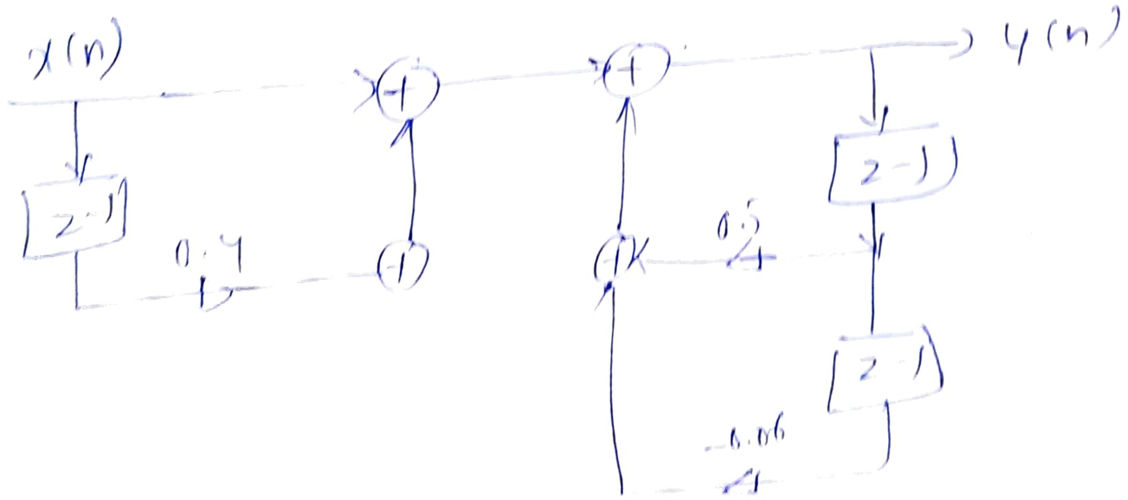
$$H(z) = \frac{1 + 0.4z^{-1}}{1 - 0.5z^{-1} + 0.06z^{-2}}$$

$$\frac{Y(z)}{X(z)} = \frac{1 + 0.4z^{-1}}{1 - 0.5z^{-1} + 0.06z^{-2}}$$

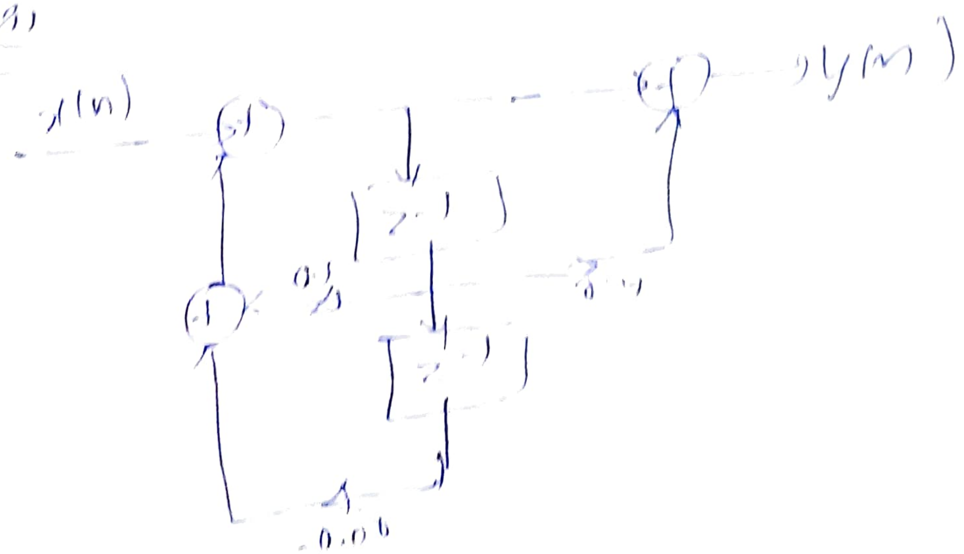
$$Y(z) - 0.5z^{-1}Y(z) + 0.06z^{-2}Y(z) = X(z) + 0.4z^{-1}X(z)$$

IZT

$$y(n) = x(n) + 0.4x(n-1) + 0.5y(n-1) - 0.06y(n-2)$$



direct form II



(2) $$H(z) = \frac{z^{-1} - 3z^{-2}}{(10 - z^{-1})(1 + 0.5z^{-1} + 0.5z^{-2})}$$

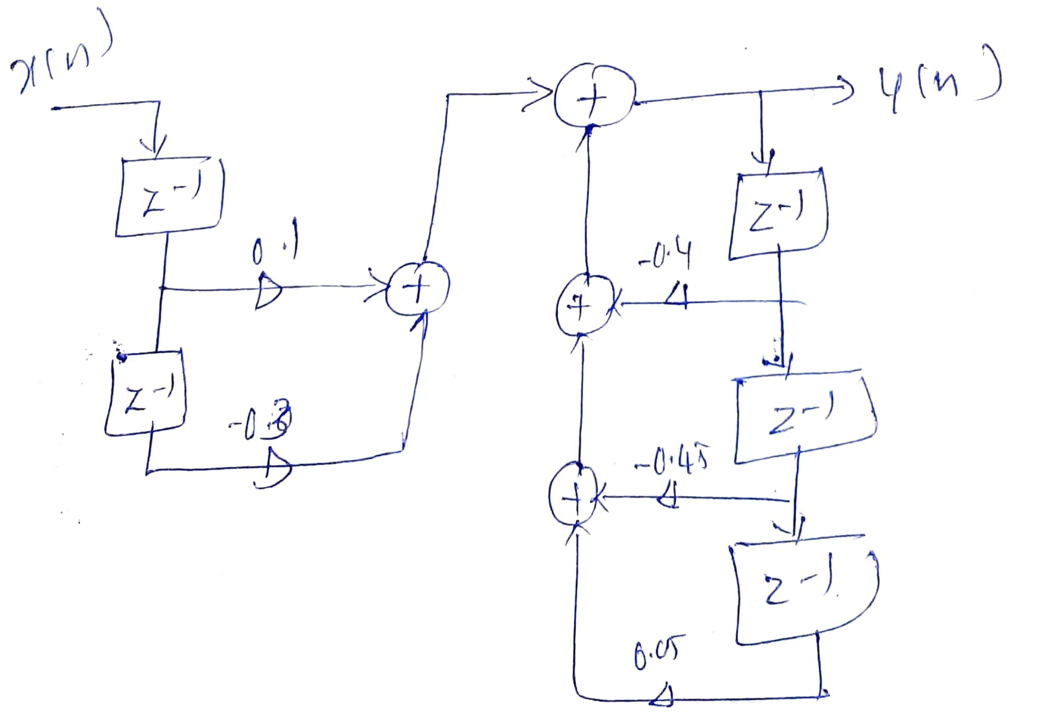
$$H(z) = \frac{z^{-1} - 3z^{-2}}{10 + 4z^{-1} + 4.5z^{-2} - 0.5z^{-3}}$$

$$\frac{Y(z)}{X(z)} = \frac{0.1z^{-1} - 0.3z^{-2}}{1 + 0.4z^{-1} + 0.45z^{-2} - 0.05z^{-3}}$$

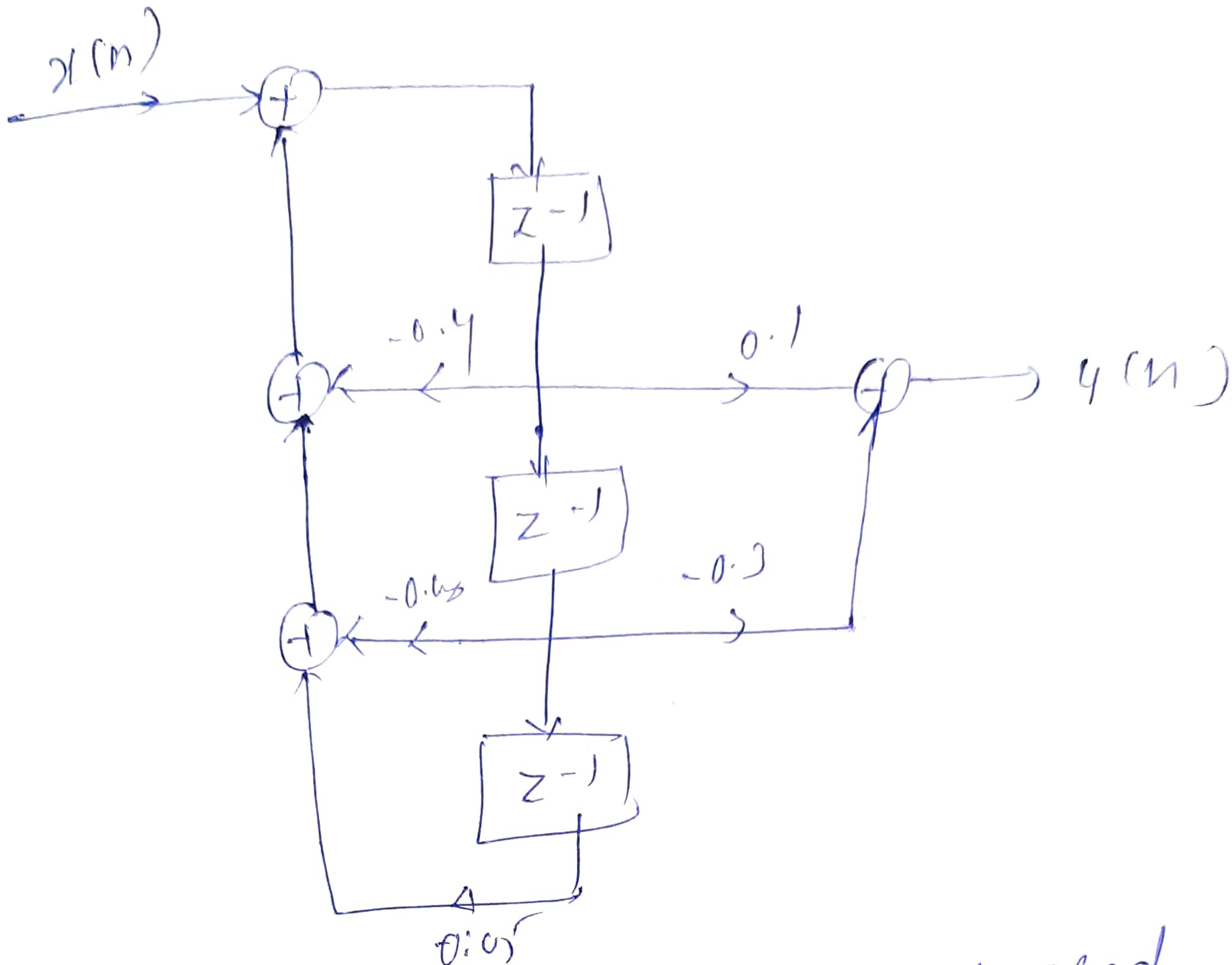
$$\begin{aligned} y(z) = {} & 0.1z^{-1}X(z) - 0.3z^{-1}X(z) \\ & - 0.4z^{-1}y(z) - 0.45z^{-2}y(z) \\ & + 0.05z^{-3}y(z) \end{aligned}$$

IZT

$$\begin{aligned} y(n) = {} & 0.1x(n-1) - 0.3x(n-2) \\ & - 0.4y(n-1) - 0.45y(n-2) \\ & + 0.05y(n-3) \end{aligned}$$



(4)



(b)

* To simulate an analog filter, the digital filter H(z) must be always be used in A/D - H(z) - D/A structure as shown in above fig

* different techniques are ~~at~~ used for designing H(z) which are listed below
  (i) Bilinear Transformation
  (ii) Impulse invariant transform
  (iii) Backward difference method
  (iv) matched z-transform method

(1) Bilinear Transformation :

Let us consider the 1st order differential eqn given by

$$a_1 y_a'(t) + a_0 y_a(t) = b_0 x_a(t) \rightarrow (1)$$

where a_1, a_0 & b_0 are arbitrary constants

Taking LT on both sides with all initial condns equal to zero

$$a_1 s Y_a(s) + a_0 Y_a(s) = b_0 X_a(s)$$

$$Y_a(s)\left[a_1 s + a_0\right] = b_0 X_a(s).$$

$$\boxed{H_a(s) \triangleq \frac{Y_a(s)}{X_a(s)} = \frac{b_0}{a_1 s + a_0} \rightarrow (2)}$$

(4)

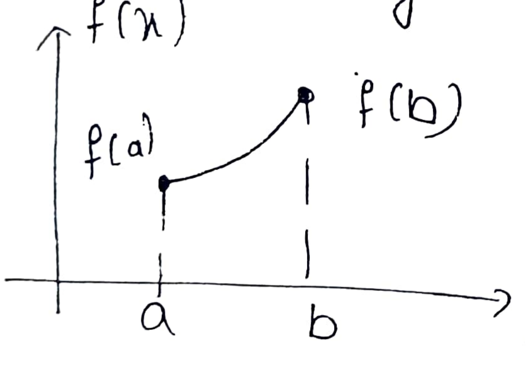
the fundamental theorem of integral calculus allows us to write

$$y_a(t) = \int_{t_0}^{t_1} y_a'(t)\,dt + y_a(t_0) \longrightarrow (3)$$

replace $t = nT$ & $t_0 = (n-1)T$ in eq (3) we get

$$y_a(nT) = \int_{(n-1)T}^{nT} y_a'(nT)\,dt + y_a(n-1)T \longrightarrow (4)$$

Recall - The trapezoidal rule of integration / .



$$\int_a^b f(x)\,dx = \left[\frac{f(a) + f(b)}{2}\right] \times (b-a)$$

$$\int_{(n-1)T}^{nT} y_a'(nT)\,dt = \left[y_a'(nT) + y_a'(n-1)T\right]$$

$$(b-a)$$

$$nT - [nT - T]$$

$$\cancel{nT} - \cancel{nT} + T$$

$$= \frac{T}{2}\left[y_a'(nT) + y_a'(n-1)T\right] \quad nT-(n-1)T$$

$\therefore$ eq (4) becomes

~~Ya(nT~~

$$y_a(nT) = \frac{T}{2}\left[y_a'(nT) + y_a'(n-1)T\right] + y_a(n-1)T \rightarrow (5)$$

from eq (1) we have

$$y_a'(t) = -\frac{a_0}{a_1} y_a(t) + \frac{b_0}{a_1} x_a(t)$$

replacing t by nT & by $(n-1)T$

we get

$$y_a'(nT) = -\frac{a_0}{a_1} y_a(nT) + \frac{b_0}{a_1} x_a(nT) \rightarrow (6)$$

$$y_a'((n-1)T) = -\frac{a_0}{a_1} y_a((n-1)T) + \frac{b_0}{a_1} x_a((n-1)T) \rightarrow (7)$$

Sub eqs (6) & (7) in (5)

$$y_a(nT) = \frac{T}{2}\left[-\frac{a_0}{a_1} y_a(nT) + \frac{b_0}{a_1} x_a(nT) - \frac{a_0}{a_1} y_a((n-1)T) + \frac{b_0}{a_1} x_a((n-1)T)\right]$$

$$+ y_a(n-1)T$$

denoting $y_a(nT)$ by $y(n)$

& $x_a(nT)$ by $x(n)$

$$y(n) = \frac{T}{2}\left\{-\frac{a_0}{a_1}y(n) + \frac{b_0}{a_1}x(n) - \frac{a_0}{a_1}y(n-1) + \frac{b_0}{a_1}x(n-1)\right\} + y(n-1).$$

taking Z-transforms on both sides, we get

$$y(z) = \frac{T}{2}\left\{-\frac{a_0}{a_1}y(z) + \frac{b_0}{a_1}x(z) - \frac{a_0}{a_1}z^{-1}y(z) + \frac{b_0}{a_1}z^{-1}x(z)\right\} + z^{-1}y(z).$$

$$= -\frac{a_0}{a_1}\frac{T}{2}y(z) + \frac{b_0}{a_1}\frac{T}{2}x(z) - \frac{a_0}{a_1}\frac{T}{2}z^{-1}y(z) + \frac{b_0}{a_1}z^{-1}\frac{T}{2}x(z) + z^{-1}y(z)$$

$$\cancel{= x(z)}$$

$$\cancel{= \frac{b_0}{a_1}\frac{T}{2}x(z)}\{1 + z^{-1}\}$$

$$\cancel{= x(z)\left\{\frac{b_0}{a_1}\frac{T}{2} + \frac{b_0}{a_1}\frac{T}{2}z^{-1}\right\} + y(z)}\left\{-\frac{a_0}{a_1}\frac{T}{2} - \frac{a_0}{a_1}\frac{T}{2}z + \frac{b_0}{a_1}z^{-1}\frac{T}{2} + z^{-1}\right\}$$

$$\cancel{= x(z)\left\{\frac{b_0T + b_0Tz^{-1}}{2a_1}\right\} + y(z)}\left\{\frac{-a_0T - a_0Tz^{-1} + b_0z^{-}}{2a_1}\right.$$

(31)

$$y(z) = \frac{b_0}{2a_1} \cdot T\{x(z) + z^{-1}x(z)\}$$

$$- \frac{a_0}{2a_1} T\{y(z) + z^{-1}y(z)\}$$

$$+ z^{-1}y(z)$$

$$y(z)\left[\frac{a_0}{2a_1} T(1+z^{-1}) - z^{-1} + 1\right]$$

$$= x(z)\left[\frac{b_0}{2a_1} T(1+z^{-1})\right]$$

$$\frac{y(z)}{x(z)} = H(z) = \frac{\frac{b_0}{2a_1} T(1+z^{-1})}{\frac{a_0}{2a_1} T(1+z^{-1}) - z^{-1} + 1}$$

$\cancel{\div (1+z^{-1})}$ $\div$ Nr & dr of RHS by $(1+z^{-1})$

$$= \frac{\frac{b_0}{2a_1} T(1)\left(\frac{1+z^{-1}}{1+z^{-1}}\right)}{\frac{a_0}{2a_1} T + \frac{(1-z^{-1})}{(1+z^{-1})}}$$

$$= \frac{\frac{T}{2a_1} b_0}{\frac{T}{2a_1}\left[a_0 + \left(\frac{1-z^{-1}}{1+z^{-1}}\right)\frac{2a_1}{T}\right]}$$

$$= \frac{b_0}{a_0 + \left\{\frac{1-z^{-1}}{1+z^{-1}}\right\} \cdot \frac{2a_1}{T}}$$

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{a_0 + a_1 \times \frac{2}{T}\left\{\frac{1-z^{-1}}{1+z^{-1}}\right\}}} \rightarrow (8)$$

comparing eq (2) & (8) we get

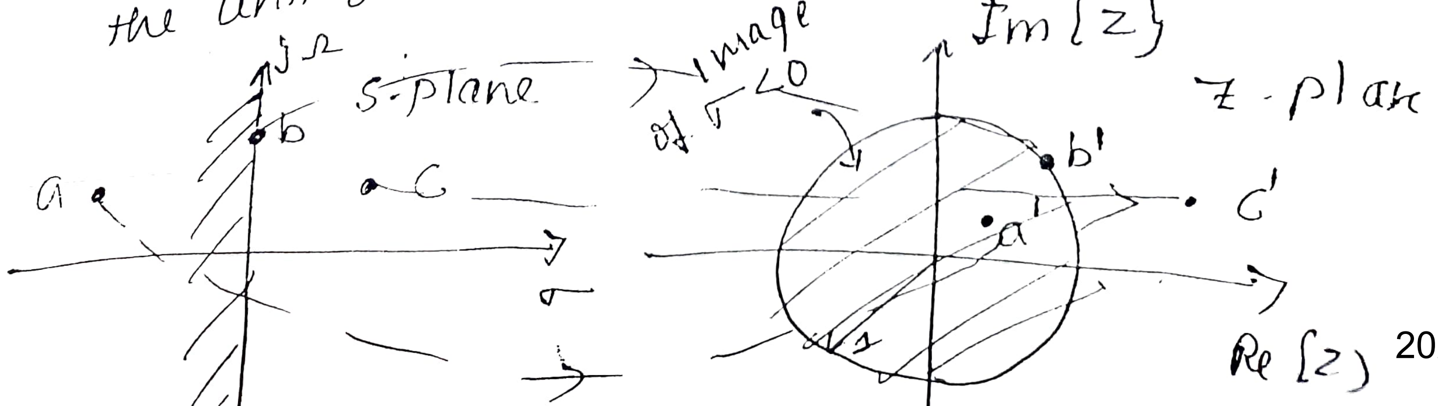
$$H(z) = H_a(s)\Big|_{s \rightarrow \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$\Rightarrow s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$$

The above eqn is called as bilinear Transformation.

<u>Properties of Bilinear ~~Trasf.~~ transformation</u>:

(1) bilinear transformation maps $j\Omega$ axis of the s-plane on to the unit circle in z-plane. Also the LHS of the s-plane is mapped inside the unit circle while the RHS ~~AA~~ of the s-plane is mapped outside the unit circle in z-plane

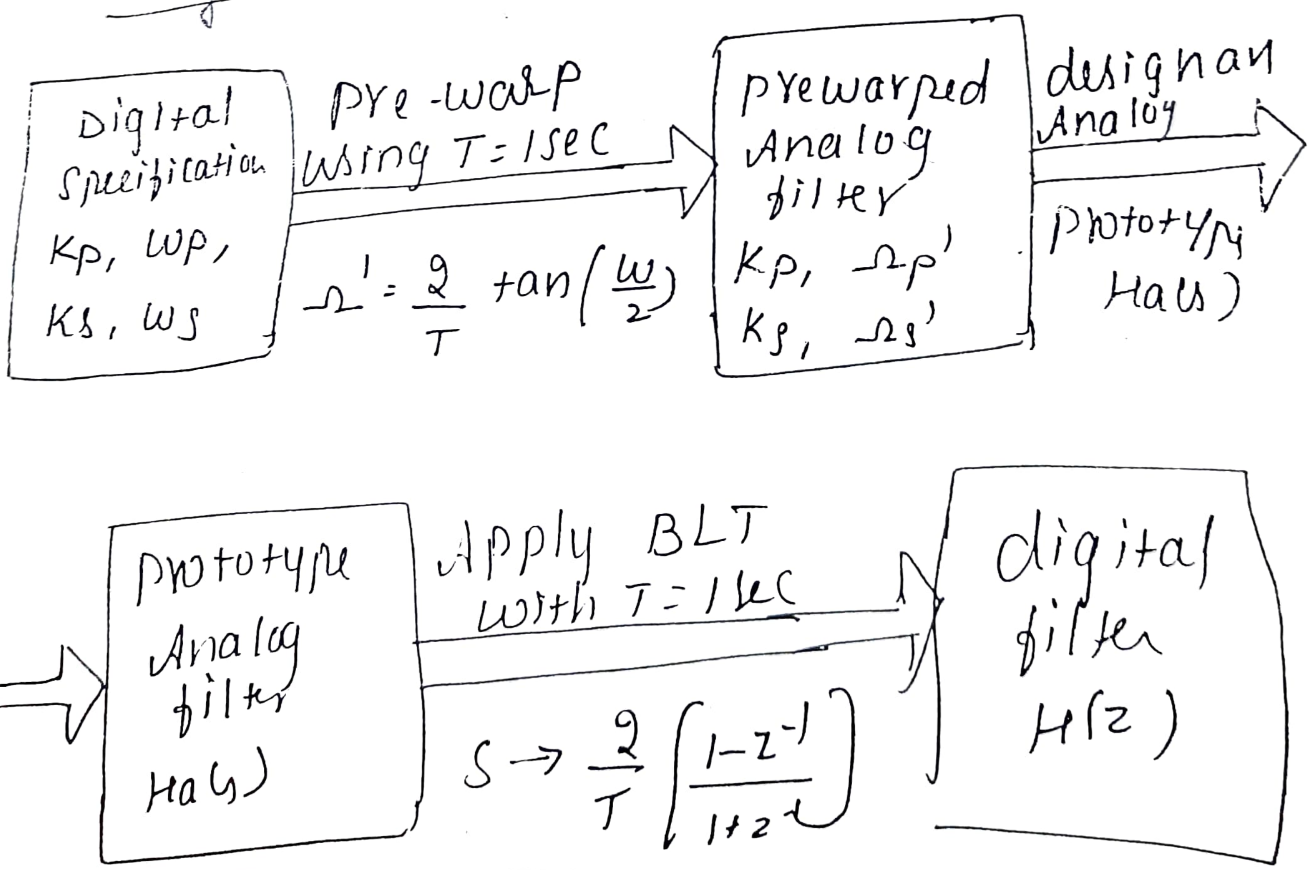


* BLT doesn't preserves the phase response, when the phase response is mapped from s-plane to z-plane

* As a consequence of this, an Analog filter having a linear phase in s-domain will no longer have a linear phase in z-domain when it is mapped using BLT.

* This is a serious Limitation of BLT.

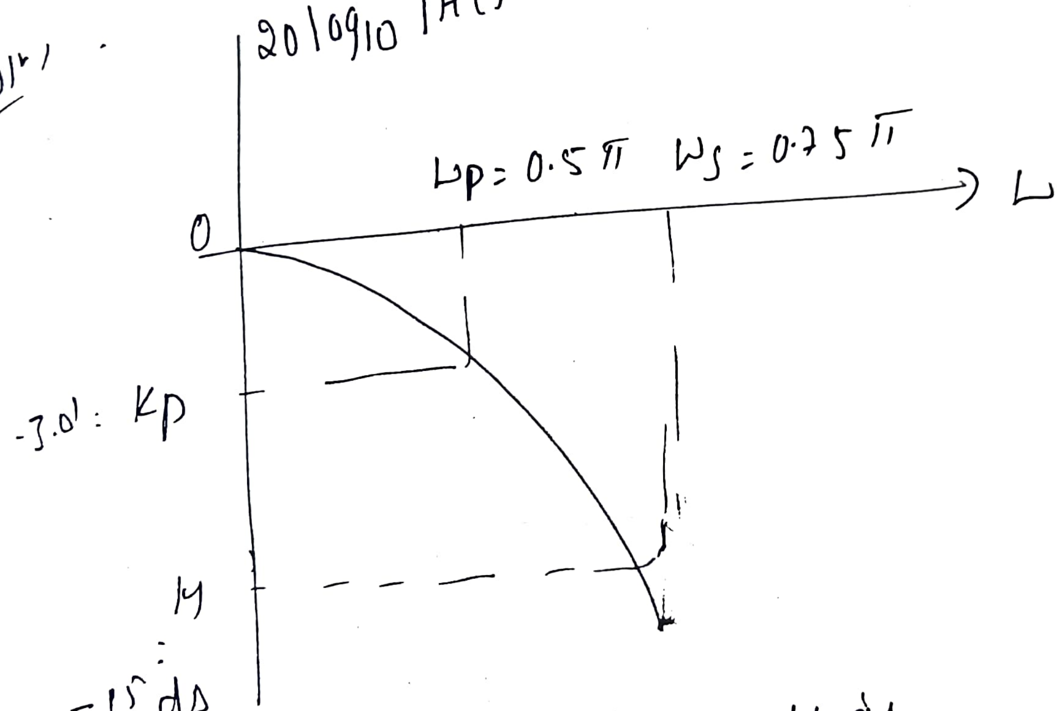
Design of Digital filters using BLT



NOTE;
(i) monotonic pass & stop bands → Butterworth
(ii) monotonic outside pass band → Chebyshev

(1) Design & realize [difference eqn] a LPF using BLT for satisfying the following specifications
(i) monotonic pass & stop band
(ii) −3.01dB cut off freq at 0.5π rad
(iii) magnitude down atleast 15dB at 0.75π rad

Sol:



Step 1:- prewarp the digital freqs

$$\Omega_p' = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right) = \frac{2}{1}\tan\left(\frac{0.5\pi}{2}\right)$$

$= 2$ rad/sec

$$\Omega_s' = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right) = 2\tan\left[\frac{0.75\pi}{2}\right]$$

(2)

$N = 1.941$

$N = 2$

(3) $H_2(s) = \dfrac{1}{s^2 + \sqrt{2}\, s + 1}$

(4)

$$\Omega c = \frac{\Omega p^1}{\left[10^{-Kp/10} - 1\right]^{1/2N}}$$

$= 2$ rad/sec

(5) $Ha(s) = H_2(s)\Big|_{s \to \frac{s}{\Omega c}} = \frac{s}{2}$

$$= \frac{4}{s^2 + 2\sqrt{2}\, s + 4}$$

(6) $H(z) = Ha(s)\Big|_{s \to \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$

$T = 1$ sec

$$= \frac{4}{4\left[\frac{1-z^{-1}}{1+z^{-1}}\right]^2 + 2\sqrt{2} \times 2 \times \left[\frac{1-z^{-1}}{1+z^{-1}}\right] + 4}$$

$$= \frac{(1+z^{-1})^2}{3.1412 + 0.5858\, z^{-2}}$$

T.F in the form of difference eqn

$$H(Z) = \frac{1 + 2z^{-1} + z^{-2}}{3.4142 + 0.5858\, z^{-2}} = \frac{Y(z)}{X(z)}$$

$(z = e^{j\omega})$

$X(z) + 2z^{-1}X(z) + z^{-2}X(z)$

$= 3.4142\, Y(z) + 0.5858\, z^{-2}\, Y(z)$

taking IZT

$3.4142\, y(n) + 0.5858\, y(n-2) = x(n) + 2x(n-1)$
$+ x(n-2)$

$\therefore y(n) = -0.1716\, y(n-2) + 0.2928\, x(n)$
$+ 0.5857\, x(n-1) + 0.2928\, x(n-2)$

(2) design a digital LPF to meet the foll specifications

(i) pass band ripple = 3 dB
(ii) pass band edge freq = 0.5π rad
(iii) min stop band atten = 15 dB
(iv) stop band atten = 0.75π rad
(v) magnitude response is monotonic outside the pass band

I $K_p = -3dB$ $K_s = -15dB$

$\omega_p = 0.5\pi$ rad $\omega_s = 0.75\pi$ rad

$\Omega_p' = \frac{2}{T}\tan\left[\frac{\omega_p}{2}\right] = 2$ rad/sec

$\Omega_s' = 4.8284$ rad/sec

II $K_p = -3dB = 20\log\left(\frac{1}{\sqrt{1+\epsilon^2}}\right)$

$$\boxed{\epsilon = 0.99763}$$

$$\delta_p = 1 - \frac{1}{\sqrt{1+\epsilon^2}} = \boxed{0.29205}$$

$K_s = -15dB = 20\log\delta_s$

$$\therefore \boxed{\delta_s = 0.17783}$$

order of the filter

$$N \geq \frac{\cosh^{-1}(1/d)}{\cosh^{-1}(1/k)}$$

$$d = \sqrt{\frac{(1-\delta_p)^{-2}-1}{(\delta_s^{-2}-1)}} = 0.180$$

$$K = \frac{\Omega_p'}{\Omega_s'} = 0.41422 \qquad N \geq 1.58$$

$$\boxed{N = 2}$$

(37)

$$a = \frac{1}{2}\left[\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right]^{1/N} - \frac{1}{2}\left[\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right]^{-1/N}$$

$$\boxed{a = 0.45601}$$

$$b = \frac{1}{2}\left[\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right]^{1/N} + \frac{1}{2}\left[\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right]^{-1/N}$$

$$\boxed{b = 1.09906}$$

$$\sigma_k = -a \sin\left[(2K-1)\frac{\pi}{2N}\right]$$

$$\Omega_k = b \cos\left[(2K-1)\frac{\pi}{2N}\right]$$

$K = 1, \ldots 2N$

$K = 1, \ldots 4$

K	σ_k	Ω_k
1	-0.32244	0.77715
2	-0.32244	-0.77715

$$V_N(s) = V_2(s) = (s-s_1)(s-s_2)$$

$$(s+0.32244-j0.7715)(s+0.32244+j0.77715)$$

$$V_2(s) = s^2 + \underbrace{0.64495}_{b_1} + \underbrace{0.70800}_{b_0}$$

$$K_N = K_2 = \frac{b_0}{\sqrt{1+\epsilon^2}} = 0.50123$$

$$H_2(s) = \frac{0.50123}{s^2 + 0.64495 + 0.7080}$$

$$H_a(s) = H_2(s) \Big|_{s \to \frac{s}{\Omega_p}},$$

$$= \frac{0.50123}{\left(\frac{s}{2}\right)^2 + 0.6449\left(\frac{s}{2}\right) + 0.7080}$$

$$\boxed{H_a(s) = \frac{2.00492}{s^2 + 1.28985 + 2.83204}}$$

$\boxed{III}$ $H(z) = H_a(s) \Big|_{s \to \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}}$

$T = 1$

$$H(z) = \frac{2.00492(1+z^{-1})^2}{2(1-z^{-1})^2 + 1.2898(1-z^{-2}) + 2.83204(1+z^{-1})}$$

Module -5

DIGITAL SIGNALPROCESSORS

Basic Architectural Features

- ✓ PDSP's should provide instructions similar to microprocessors.
- ✓ Basic computational capabilities provided by the way of instructions should include the following:
    1. Arithmetic operations
    2. logical operations
    3. MAC operations
    4. Signal scaling operations
- ✓ To perform all these operations a dedicated high speed H/W must be provided.
- ✓ The architecture should include the following H/W features also:

1. on chip registers – storage of intermediate results
2. on chip memories – signal samples (RAM)
3. on chip pgm memory – pgms & fixed data such as filter coefficients (ROM).

1. Investigate the basic features that should be provided in DSP architecture to be used to implement the following Nth order FIR filter.

$$Y(n) = \sum_{i=0}^{N-1} h(i)\, x(n-i) \; ; \qquad n = 0,1,2\ldots.$$

Where x(n) denotes the i/p samples
y(n) the o/p samples
x(n-i) is the i/p sample I samples earlier than x(n)
and h(i) the i^{th} filter coefficients

DSP Computational Building Blocks

- The basic building blocks that are essential to carry out DSP computations are as follows :
  1. multiplier
  2. shifter
  3. MAC unit
  4. ALU

MULTIPLIER

- ✓ Earlier multiplication schemes relied either on
  S/W or Micro coded controllers
- ✓ Both these options require several processor cycles to complete the multiplication
- ✓ The advances made in VLSI technology in speed & size made possible the H/W implementation of parallel multipliers

- ✓Before designing an actual multiplier, the specifications such as speed , accuracy and dynamic range must be clear.

- ✓accuracy and dynamic range – is decided based on the number of bits used to represent the multiplication operands and whether they are represented in fixed point or floating point format.

- ✓Speed – is decided by the architecture employed.

Parallel Multiplier

Let us consider the multiplication of 2 unsigned numbers A & B

Let

A --- represents m bits multiplicand [Am-1, Am-2…A0]

B --- represents n bits multiplier [Bn-1, Bn-2…B0]

P ---product of A & B . Max (m+n) bits

$$A = \sum_{i=0}^{m-1} A_i 2^i \qquad \text{----------------------1}$$

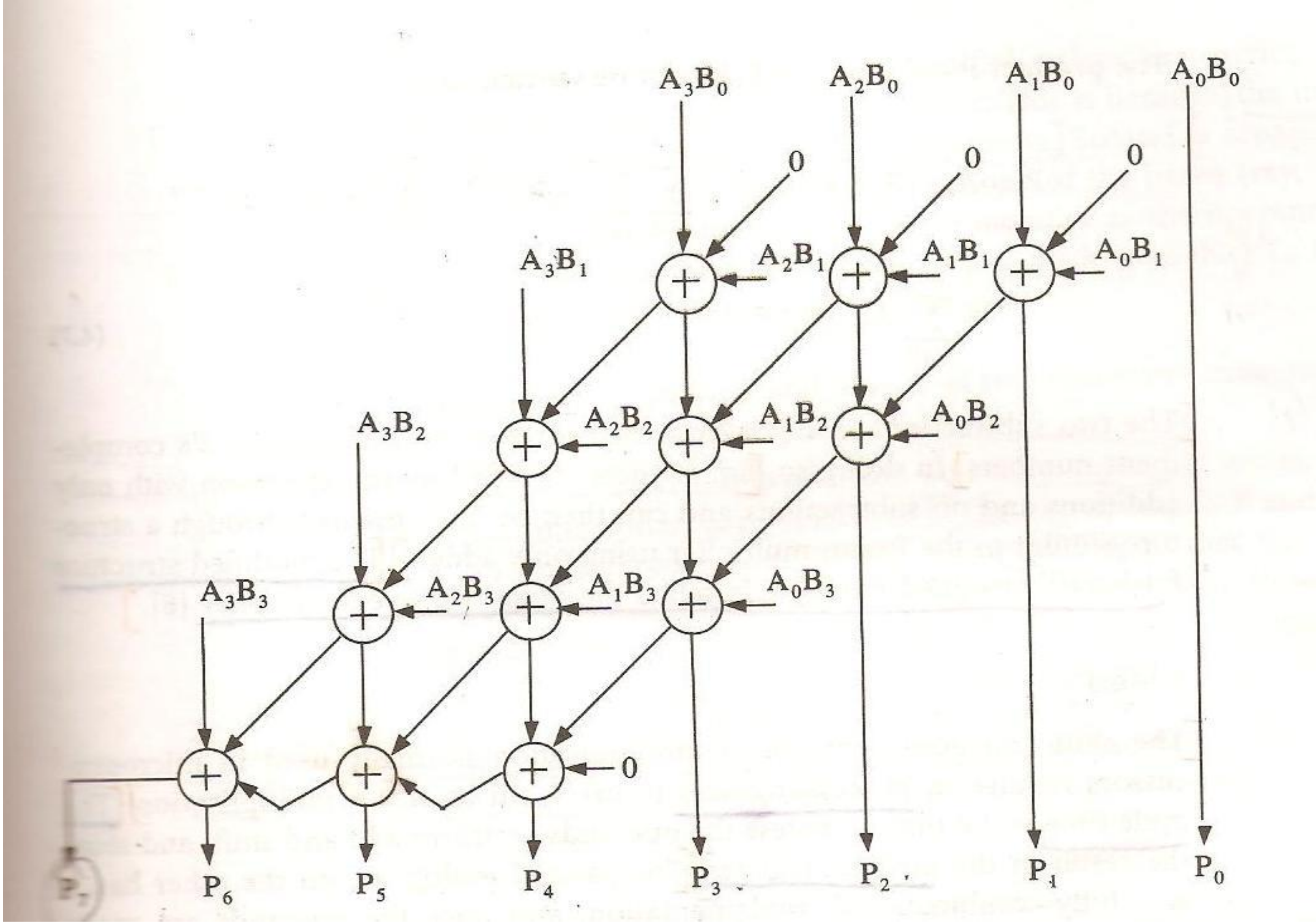
$$B = \sum_{j=0}^{n-1} B_j 2^j \qquad \text{---------------------2}$$

$$p = \sum_{j=0}^{n-1} \left[\sum_{i=0}^{m-1} A_i B_j 2^{i+j} \right] \qquad \text{-----3}$$

The multiplication operations using 4 bit s for A & B are shown below.

				A3	A2	A1	A0
				B3	B2	B1	B0
				A3BO	A2B0	A1B0	A0B0
			A3B1	A2B1	A1B1	A0B1	
		A3B2	A2B2	A1B2	A0B2		
	A3B3	A2B3	A1B3	A0B3			
P7	P6	P5	P4	P3	P2	P1	P0

The fig below shows the H/W structure of the multiplier for this example and is called as Braun multiplier.



The structure of 4 × 4 Braun multiplier

- For a $n \times n$ multiplier we require $n(n-1)$ adders.
- The Structure requires : 12 [3 i/p & 2 o/p]
- Braun's multiplier does not take in account the signs of the numbers that are being multiplied.
- Additional H/W is required before & after the multiplication when signed numbers represented in 2's complement are used
- Let us consider A& B represented in 2's complement format

A & B having m & n bits respectively

$$A = -A_{m-1}2^{m-1} + \sum_{i=0}^{m-2} A_i 2^i \quad \text{-------1}$$

$$B = -B_{n-1}2^{n-1} + \sum_{j=0}^{n-2} B_j 2^j \quad \text{-----2}$$

The product $p = p_{m+n-1}$ is written as

$$p = A_{m-1}B_{n-1}2^{m+n-2} + \sum_{i=0}^{m-2}\sum_{j=0}^{n-2} A_i B_j 2^{i+j} \quad \text{-----3}$$

- the 2 subtractions in eqn 3 can be expressed as addition of 2's complement number.
- This can be implemented through a structure similar to the Braun multipliers using only adders.
- This modified structure is called as Baugh Wooley multiplier.

Baugh wooley multiplier is shown in Fig. 2.22.

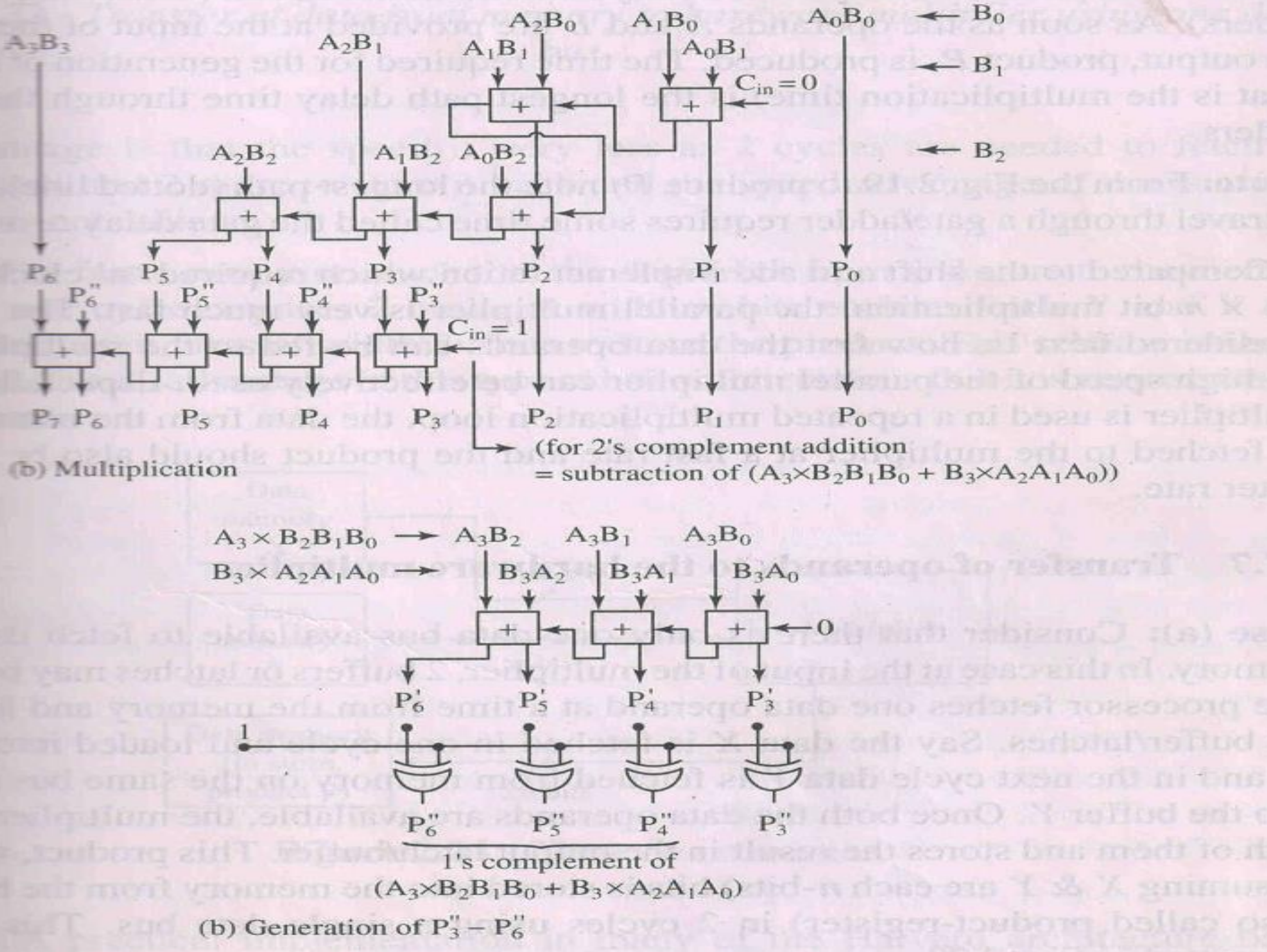


Figure 2.22 *Structure of Baugh wooley multiplier for signed number multiplication of two 4-bit signed numbers.*

shifters

- Is an essential component of any DSP architecture.

- Required to scale down or scale up the operands & results to avoid errors resulting from overflows & underflows during computation.

Let us consider the following cases.

1. Let us consider the sum of ‘N’ numbers each represented by n bits.

- As accumulated sum grows the number of bits required to represent it also increases

- The max no of bits to which the sum can grow is $(n + \log_2 N)$ bits

- If each of the N no is scaled down by $\log_2 N$ bits prior to addition, the loss of the result due to overflow can be avoided.

- The accumulator will then hold the sum scaled down by $\log_2 N$ bits.

- Accuracy of sum is lost but the summation would be completed with out overflow error.

- The actual sum can be obtained by scaling up the result by $\log_2 N$ bits when required.

2. When 2 no’s each of ‘n’ bits are multiplied, the product can have a max of ‘2n’ bits

- when this product is saved in memory which is ‘n’ bit wide, the lower order ‘n’ bits are generally discarded resulting in loss of accuracy
- in case of multiplication of 2 signed no’s, the accuracy can be slightly improved
- By shifting the product by one bit position to the left before saving ‘n’ higher order bits.
- The accuracy improves because instead of discarding n dits we now discard (n-1) bits

3. When carrying out floating point additions

- the operands should be normalized to have same exponent.
- This is done by shifting one of the operands by required no of bit positions

Problems:

1.It is required to find the sum of 64 no's each represented by 16 bits. How many bits should the accumulator have so that the sum can be computed with out the occurrence of overflow error or loss of accuracy?

Ans:- sum grows by a max of $\log_2 64 = 6$ bits

to avoid over flow the number of bits the accumulator should have is $16+6 = 22$

2. If for problem 1 it is decided to have an accumulator with only 16 bits but shift the numbers before addition to prevent overflow . By how many bits should each number be shifted?

By 6 bits to right since the sum grows by 6 bits

3. If all the numbers in problem 2 are fixed point integers , what is the actual sum of the numbers

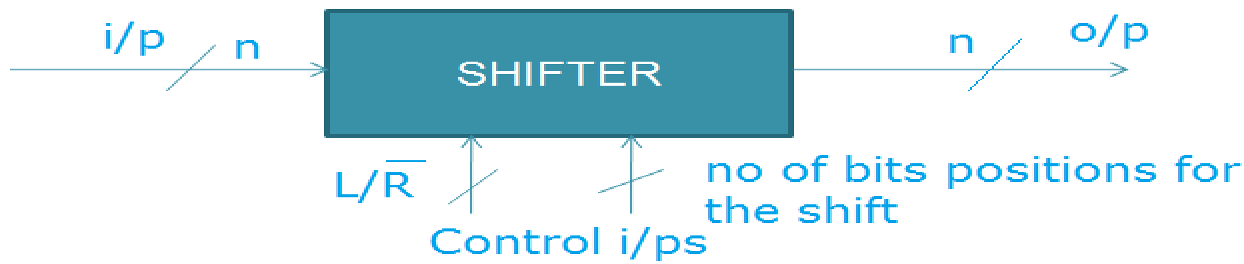
The actual sum = (contents of accumulator) $\times 2^6$

4. What is the error in computation of the sum in problem 3

Since 6 lower bits are lost the sum could be off by as much as $2^6 - 1 = 63$

Barrel Shifter

- In µps shifting is implemented by a operation similar to one performed in a shift register .
- The operation takes one clock cycle for every single bit shift.
- In DSP's shifting of several bits in a single cycle is possible by a combinational ckt known as Barrel Shifter.
- Barrel shifter connects the i/p lines representing a word to a group of output lines with the required shift which is determined by its control inputs as shown below
- Control i/ps also determines the direction of the shift (left or right)



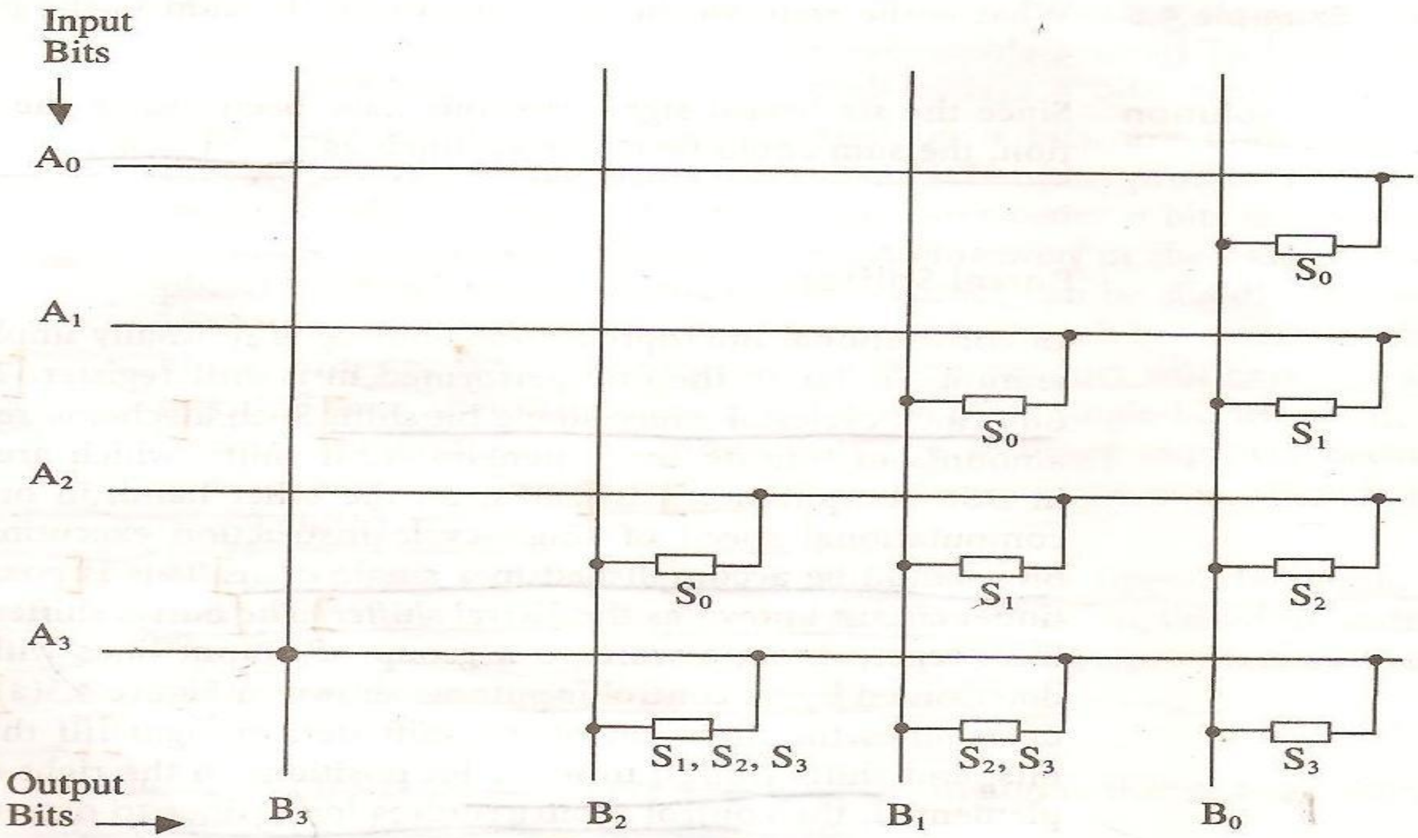
B.D of barrel shifter

- If the input word has 'n' bits, shift is from 0 to n-1 bits

- The control i/p requires $\log_2 n$ lines to determine the no. of bits to be shifted.
- One more line is required to indicate the direction of the shift.

- Left shift:- bits shifted out of the i/p word are discarded and new bits positions are filled with zeros.

- Right shift:- the new bit positions are replicated with the MSB to maintain the sign of the shifted result.



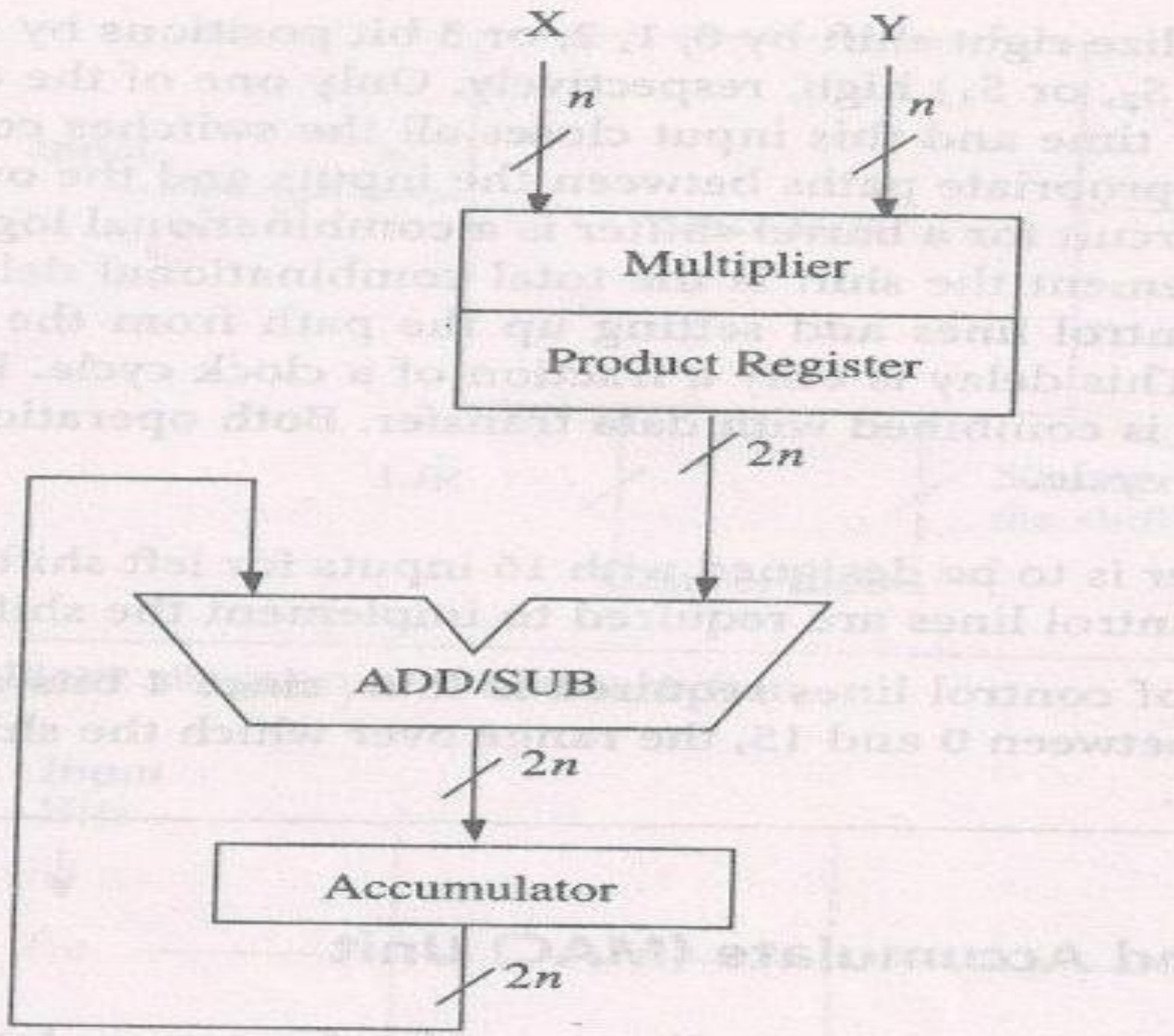
Input	Shift (Switch)	Output ($B_3B_2B_1B_0$)
$A_3A_2A_1A_0$	0 (S_0)	$A_3A_2A_1A_0$
$A_3A_2A_1A_0$	1 (S_1)	$A_3A_3A_2A_1$
$A_3A_2A_1A_0$	2 (S_2)	$A_3A_3A_3A_2$
$A_3A_2A_1A_0$	3 (S_3)	$A_3A_3A_3A_3$

1. A barrel shifter is to designed with 16 inputs for left shift form 0 to 15 bits. How many control lines are required to implement the shifter .

ANS: 4 control lines are required.

Multiply & Accumulate (MAC) Unit

- The configuration of a multiply & accumulate unit is commonly known as MAC unit as shown below.
- The MAC unit is used to implement functions of the type A +BC.
- Multiplication & accumulation are 2 distinct operations, each normally require separate instruction execution cycle.
- But they can work in parallel.


X
Y
n
n
Multiplier
Product Register
2n
ADD/SUB
2n
Accumulator
2n

A MAC unit

Pipelined operation of MAC

- The pipelined operation makes use of the fact that the multiplier and adder can work separately and simultaneously.

- Consider a example

$$y(n) = x_1h_1 + x_2h_2 + x_3h_3$$

Cycle time number	Multiplier	Adder
1	x_1h_1	-
2	x_2h_2	$A+x_1h_1 \rightarrow A$
3	x_3h_3	$A+x_2h_2 \rightarrow A$
4	-	$A+x_3h_3 \rightarrow A$

- At a time when the multiplier is computing the product , the accumulator accumulates the product of the previous multiplication.
- if N products are to be accumulated, N-1 multiplies overlap with accumulations.
- During every first multiply, the accumulator is idle since there is nothing to accumulate.
- Like wise during the very last accumulation, the multiplier is idle since all the ‘N’ products have been completed.
- It takes a total number of ‘N+1’ instruction execution cycles to complete the sum of products of ‘N’ multiplications.
- If N is large the pipelined operation of multiplier & accumulator work in parallel to execute a MAC operation per cycle.

1. If a sum of 256 products is to be computed using a pipelined MAC unit, & if the MAC execution time of the unit is 100 nsec, what will be the total time required to complete the operations?

ANS: total time required = $257 \times 100 \times 10^{-9}$ sec =25.7µsec.

Over flow & under flow

When designing a MAC attention has to be paid for

1. word size at the i/p of the multiplier
2. the sizes of the add/ subtract unit &
3. The accumulator

as over flow & under flow conditions can be encountered .

- Techniques to prevent over flow & under flow conditions are

1. Barrel shifters at i/p & o/ps of MAC unit
2. Guard bits in the accumulator
3. Saturation logic

1. <u>Barrel shifters</u>

- shifters are normally provided at i/p & o/ps of MAC unit.
- the i/p shifters helps to normalize the data samples as they are fed to the multiplier.
- The shifter at the output are used to denormalize the result after the sum of products before storing in memory.
- o/p shifters can also be used to discard the redundant sign bits or to shift the o/p by required no. of positions.

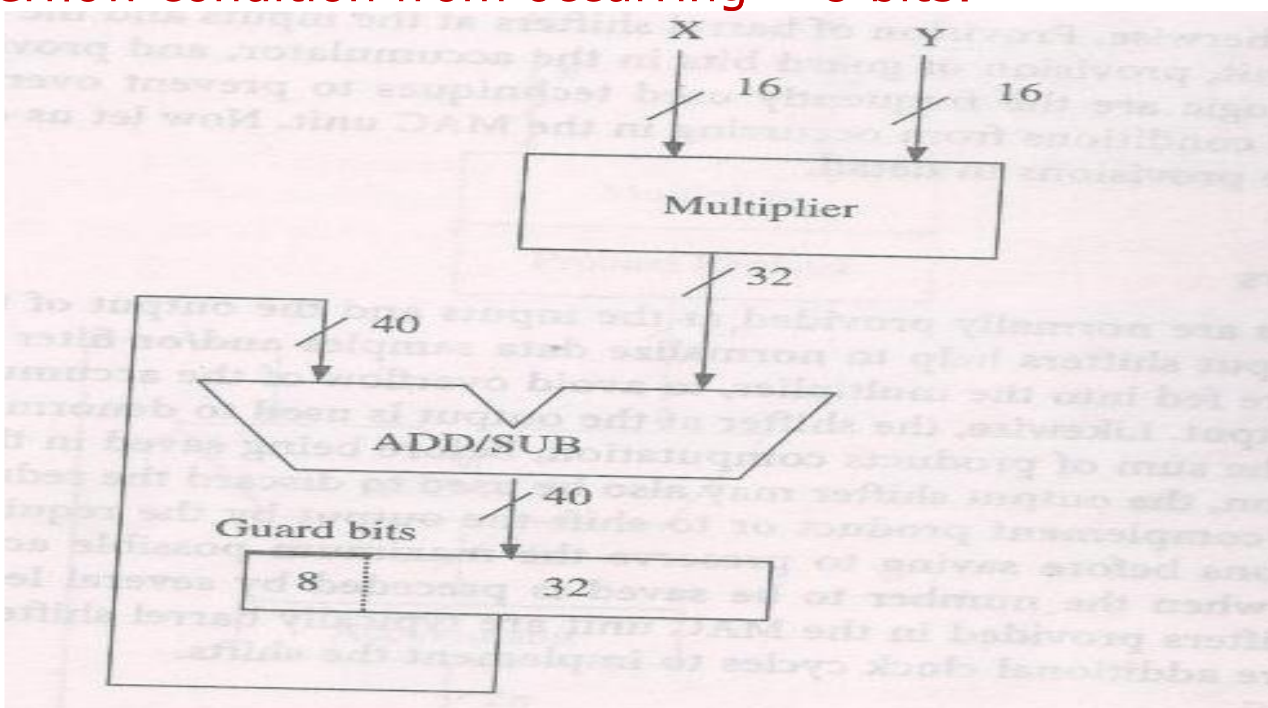
2. Guard bits:

- If accuracy is preserved, the i/ps are not normalized.
- When repetitive MAC operations are performed the accumulator sum grows with each MAC operation.
- This increases the no. of bits required to present the result w/o loss of accuracy.
- So extra bits must be provided in accumulator called as guard bits or extension bits.
- After completion of computation, the required sum of product , the extension bits may be saved as a separate word if required.
- Or. The sum along with the guard bits may be shifted by required amount & be saved as a single word.

2. consider a MAC units whose inputs are 16 bits nos. if 256 products are to be summed up in this MAC , how many guard bits should be provided for the accumulator to prevent overflow condition from occurring.

Ans: 16 ×16 multiplication has 32 bits. Since 256 such products are summed, the sum can grow by max of $\log_2 256 = 8$ bits

guard bits should be provided for the accumulator to prevent overflow condition from occurring = 8 bits.



3. Saturation logic: -

- With or w/o guard bits ,an over flow condition occurs when the accumulated result becomes larger than the largest no. it can hold.

- When handling a –ve no. an under flow will occur if the accumulator becomes smaller than the smallest no. it can hold.

- So it is better to limit the accumulator contents to most +ve (most –ve) to avoid an error known as wrap around error.

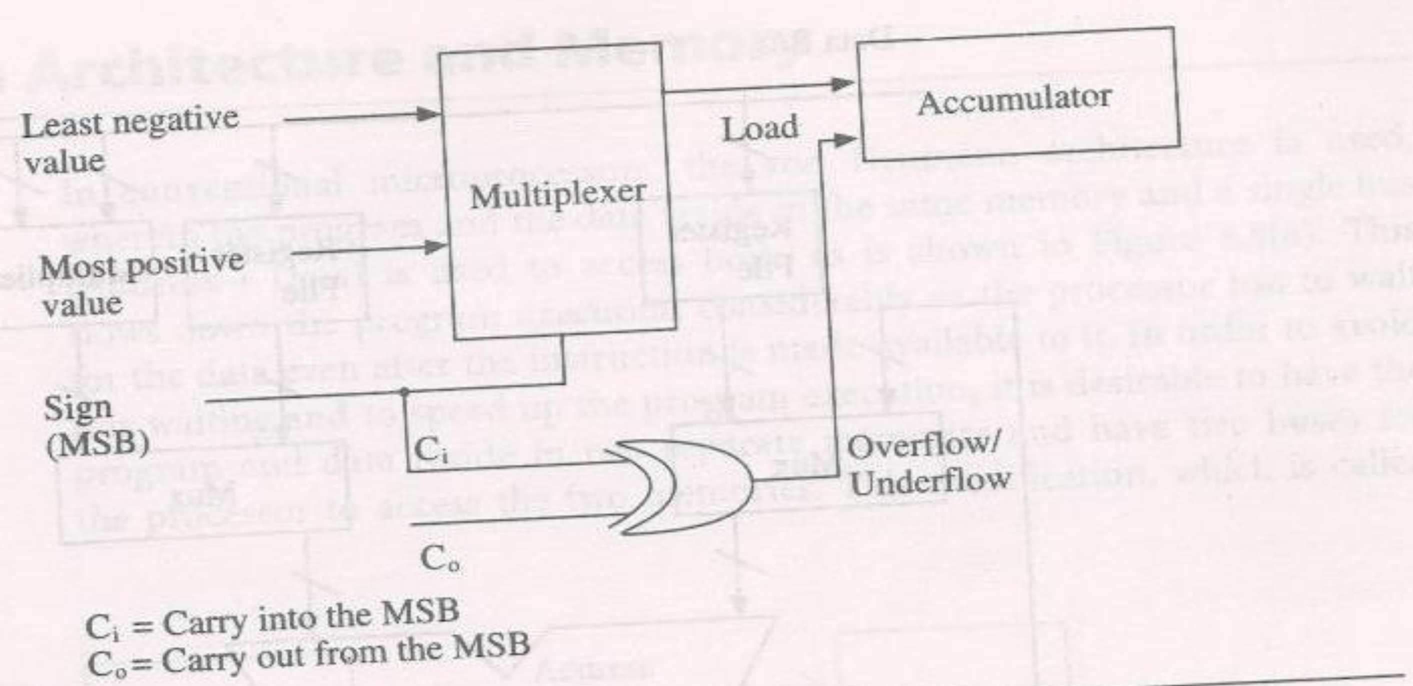
- Limiting the accumulator contents to its saturation limit is achieved with a simple logic ckt called as saturation logic as shown below.

- This ckt detects the overflow / underflow condition & accordingly loads the accumulator with to most +ve or most –ve value

- the overflow / underflow condition is detected by monitoring the carry into the MSB & the carry out of MSB.

- If carry-in ≠ to carry out , the overflow / underflow occurs

- The selection bet' the most +ve or most –ve value is made based on the sign bit of the no.



A schematic diagram of the saturation logic

Module-1

Chapter 3 in Text

COMPUTATIONAL ACCURACY IN DSP APPLICATIONS

- Number formats for signals & coefficients in DSP systems:-

Conditions/ parameters to represent Numbers:-

- Range
- precission of signals
- coefficients to be represented
- Hardware complexity
- Speed Requirements

FORMATS

(i) Fixed-point format

Problem: What is the range of numbers that can be represented in a fixed-point format using 16 bits if the numbers are treated as (a) signed integers, (b) signed fractions?

(A) $n=16$

(a) Range:- -2^{n-1} to $2^{n-1}-1$

-2^{15} to $2^{15}-1$

$= -32768$ to 32767

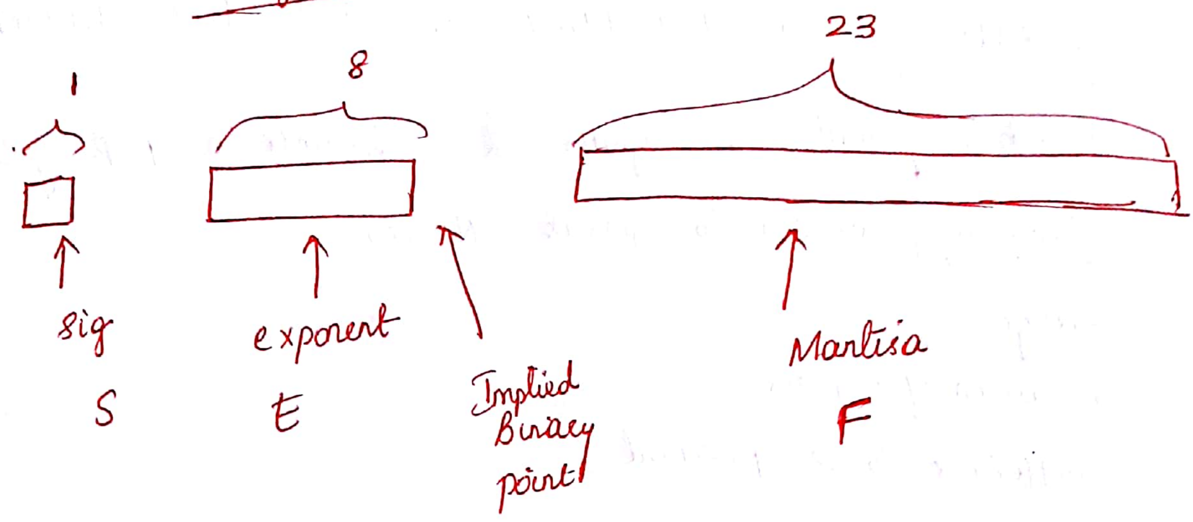
(b) Range $=$ -1 to $(1-2^{-(n-1)})$

-1 to $(1-2^{-15})$

-1 to $+.999969482$

(ii) Double precision fixed-point format

a) Floating-point format



Let M_x – mantissa

E_x – Exponent

value of $\boxed{x = M_x \; 2^{E_x}}$

Consider 2 floating point numbers x & y

then product

$$xy = M_x M_y \; 2^{E_x + E_y}$$

The value represented by the data format

$$x = (-1)^S * 2^{(E - bias)} * 1.F$$

F – magnitude fraction of the mantissa

E – integer

S – Signed bit

Bias – depends upon the bits reversed for the exponent.

$$\text{Range} = -(2 - 2^{-n}) \text{ to } +(2 - 2^{-n})$$

Ex) Find the decimal equivalent of the floating-point binary number 1011000011100. Assume a format similar to IEEE-754 in which the MSB is the sign bit followed by 4 exponent bits followed by 8 bits for the fractional part.

(A) Since MSB = 1

The Number is Negative

$$F = \begin{array}{c} 10110|00011100 \\ S\ 11\ 10\ 9\ 8|7\ 6\ 5\ 4\ 3\ 2\ 1\ 0 \\ E \qquad F \end{array}$$

$$\begin{array}{c|cccc|cccccccc} & 4 & 3 & 2 & 1 \; 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{array}$$

$$F = 2^{-4} + 2^{-5} + 2^{-6}$$

$$= .109375$$

$$E = 2^1 + 2^2 = 6$$

∴ Value of Number is

$$x = (-1)^S \times 2^{(E-bias)} * 1.F$$

$$x = (-1)^1 * 2^{(6-7)} * 1.109375$$

$$x = -0.5546875$$

Bias - Max positive Number in E bits

E = 4 bits

$$\begin{array}{c|c|c|c} 3 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}$$

+ve $2^2 + 2^1 + 2^0$

$= 7$

(Q). Using 16 bits for the mantissa and 8 bits for the exponent, what is the range of numbers that can be represented using the floating point format similar to IEEE-754?

(A) The most -ve number will have its mantissa $= -2 + 2^{-16}$ ∴ $(-2 + 2^{-n})$

$$\text{Exponent} = (2^{n} - 1) - (2^{n-1} - 1)$$

$n = 8$

$$= 255 - 127$$

$$= 128$$; -ve No = $-1.999984741 \times 2^{128}$

The most +ve ~~Number~~ Number is

$$+1.999984741 \times 2^{128}$$

4

Block Floating Point Format

increases the range & precision of a given fixed point format by retaining as many lower-order bits as is possible.

Q) The following 12-bit Binary fractions are to be stored in an 8-bit memory. Show how they can be represented in Block Floating point format so as to improve accuracy

000001110011

000011110000

000000111111

000010101010

(A) Using 8-bit fixed point format

$\boxed{0000}$ 0111 / 0011

$\boxed{0000}$ 1111 / 0000

$\boxed{0000}$ 0011 / 1111

$\boxed{0000}$ 1010 / 1010

Last 4 bits will be discarded.

But in all 4 numbers, there are 4 leading zeros

so

01110011×2^{-4}

11110000×2^{-4}

00111111×2^{-4}

10101010×2^{-4}

Dynamic Range & Precision

The Dynamic range of a signal is the ratio of the maximum value to the minimum value that the signal can take in the given number representation scheme.

→ Dynamic Range ∝ No. of Bits used to represent
→ Increases by 6dB for every additional bit.

Resolution is the minimum value that can be represented using a number representation format.

Consider N bits

Resolution $= 1/2^N$ for large N

Precision is an issue related to the speed of the DSP implementation.

Q) Calculate the dynamic range and precision of each of the following Number representation formats

(a) 24 bit, single precision, fixed point format

Since each bit gives a dynamic range of 6 dB, total dynamic Range is $24 \times 6 = 144$ dB

Percentage resolution is $1/2^{24} \times 100 = \underline{6 \times 10^{-6}}\ \%$

(b) 48-bit, double-precision, fixed point format

Each Bit gives dynamic range of 6 dB

total dynamic range $= 48 \times 6 = \underline{288\text{ dB}}$

Percentage resolution is $1/2^{48} \times 100 = 4 \times 10^{-13}\ \%$

(c) a floating point format with a 16-bit mantissa & 8bit exponent.

For 8 bit exponent bits, dynamic range is $(2^8-1) \times 6 = \underline{1530\text{ dB}}$

% resolution depends on 16 bit mantissa

$$\left(1/2^{16}\right) \times 100 = \underline{1.5 \times 10^{-3}\ \%}$$